

“Almost-stable” matchings in the Hospitals / Residents problem with Couples

David F. Manlove*, Iain McBride** and James Trimble*

School of Computing Science, Sir Alwyn Williams Building,
University of Glasgow, Glasgow G12 8QQ, UK

Abstract. The Hospitals / Residents problem with Couples (HRC) models the allocation of intending junior doctors to hospitals where couples are allowed to submit joint preference lists over pairs of (typically geographically close) hospitals. It is known that a stable matching need not exist, so we consider MIN BP HRC, the problem of finding a matching that admits the minimum number of blocking pairs (i.e., is “as stable as possible”). We show that this problem is NP-hard and difficult to approximate even in the highly restricted case that each couple finds only one hospital pair acceptable. However if we further assume that the preference list of each single resident and hospital is of length at most 2, we give a polynomial-time algorithm for this case. We then present the first Integer Programming (IP) and Constraint Programming (CP) models for MIN BP HRC. Finally, we discuss an empirical evaluation of these models applied to randomly-generated instances of MIN BP HRC. We find that on average, the CP model is about 1.15 times faster than the IP model, and when presolving is applied to the CP model, it is on average 8.14 times faster. We further observe that the number of blocking pairs admitted by a solution is very small, i.e., usually at most 1, and never more than 2, for the (28,000) instances considered.

1 Introduction

The Hospitals / Residents problem. The *Hospitals / Residents problem* (HR) [14] is a many-to-one allocation problem that models the assignment process involved in centralised matching schemes such as the National Resident Matching Program (NRMP) [45] which assigns graduating medical students to hospital posts in the USA. Analogous schemes exist in Canada [40] and Japan [42]. A similar process was used until recently to match medical graduates to Foundation Programme places in Scotland: the Scottish Foundation Allocation Scheme (SFAS) [21]. Moreover, similar matching schemes exist in the context of Higher Education admission in Hungary [4,43], Spain [32], Turkey [3] and Ireland [41,43]. The reader is referred to [43] for details of matching practices in a number of practical contexts throughout Europe.

* Supported by Engineering and Physical Sciences Research Council grants EP/K010042/1 and EP/N508792/1. Email david.manlove@glasgow.ac.uk.

** Supported by a SICSA Prize PhD Studentship.

An instance of HR consists of two sets of agents – a set $R = \{r_1, \dots, r_{n_1}\}$ containing *residents* and a set $H = \{h_1, \dots, h_{n_2}\}$ containing *hospitals*. Every resident expresses a linear preference over some subset of the hospitals, his *preference list*. The hospitals in a resident's preference list are his *acceptable* partners; all other hospitals being *unacceptable*. Every hospital expresses a linear preference over those residents who find it acceptable. Further, each hospital $h_j \in H$ has a positive integral *capacity* c_j , the maximum number of residents to which it may be assigned. A *matching* M is a set of acceptable resident-hospital pairs such that each resident appears in at most one pair and each hospital h_j belongs to at most c_j pairs. If $(r_i, h_j) \in M$ then r_i is said to be *assigned* to h_j , $M(r_i)$ denotes h_j , and r_i is an *assignee* of h_j . Given $r_i \in R$, if r_i does not belong to any pair in M then r_i is said to be *unassigned*. Given $h_j \in H$, we let $M(h_j)$ denote the set of assignees of h_j in M . Hospital h_j is *undersubscribed*, *full* or *oversubscribed* according as $|M(h_j)|$ is less than, equal to, or larger than c_j , respectively.

Roth [34] argued that a key property to be satisfied by any matching M in an instance I of HR is *stability*, which ensures that M admits no *blocking pair* in I . Informally, such a pair comprises a resident r_i and a hospital h_j , both of whom have an incentive to disregard their assignments (if any) and become matched to one another outside of M , undermining its integrity. A matching is *stable* if it admits no blocking pair. It is known that every instance of HR admits at least one stable matching, which can be found in time linear in the size of the instance [14].

The Hospitals / Residents problem with Couples. The Hospitals / Residents problem with Couples (HRC) is a generalisation of HR that is important in practical applications because it models the case where some of the residents may apply jointly in couples, so that they may be matched to hospitals that are geographically close to one another. In order to ensure this, a couple submits a joint preference list over pairs of hospitals, rather than individual hospitals. Matching schemes for junior doctors such as the NRMP [45] allow couples to apply jointly, as do assignment processes in the US Navy [31, 37, 39] (for which HRC is an appropriate problem model), for example.

Formally, an instance I of HRC consists of a set $R = \{r_1, \dots, r_{n_1}\}$ containing *residents* and a set $H = \{h_1, \dots, h_{n_2}\}$ containing *hospitals*. The residents in R are partitioned into two sets, S and S' . The set S consists of *single* residents and the set S' consists of those residents involved in *couples*. There is a set $C = \{(r_i, r_j) : r_i, r_j \in S'\}$ of *couples* such that each resident in S' belongs to exactly one pair in C .

Each single resident $r_i \in S$ expresses a linear preference order over some subset of the hospitals, his *acceptable* hospitals; all other hospitals being *unacceptable*. Each couple $(r_i, r_j) \in C$ expresses a joint linear preference order over a subset A of $H \times H$ where $(h_p, h_q) \in A$ represents the simultaneous assignment of r_i to h_p and r_j to h_q . The hospital pairs in A represent those joint assignments that are *acceptable* to (r_i, r_j) , all other joint assignments being *unacceptable*. Each hospital $h_j \in H$ expresses a linear preference order over those residents

who find it acceptable, either as a single resident or as part of a couple, and as in the case of HR, each hospital $h_j \in H$ has a positive integral *capacity* c_j .

A *matching* M in I is defined as in HR case, with the additional restriction that, for each couple $(r_i, r_j) \in C$, either both r_i and r_j appear in no pair of M , or else $\{(r_i, h_k), (r_j, h_l)\} \subseteq M$ for some pair (h_k, h_l) that (r_i, r_j) find acceptable. In the former case, (r_i, r_j) are said to be *unassigned*, whilst in the latter case, (r_i, r_j) are said to be *jointly assigned* to (h_k, h_l) . Given a resident $r_i \in R$, the definitions of $M(r_i)$, *assigned* and *unassigned* are the same as for the HR case, whilst for a hospital $h_j \in H$, the definitions of *assignees*, $M(h_j)$, *undersubscribed*, *full* and *oversubscribed* for hospitals are also the same as before.

We seek a *stable* matching, which guarantees that no resident and hospital, and no couple and pair of hospitals, have an incentive to deviate from their assignments and become assigned to each other outside of the matching. Roth [34] considered stability in the HRC context but did not define the concept explicitly. Whilst Gusfield and Irving [17] gave a formal definition of a blocking pair, it neglected to deal with the case that both members of a couple may wish to be assigned to the same hospital. A number of other stability definitions for HRC have since been given in the literature that address this issue (see [6] and [23, Section 5.3] for more details), including that of McDermid and Manlove [27], which we adopt in this paper. We repeat their definition again here for completeness.

Definition 1 ([27]) *Let I be an instance of HRC. A matching M is stable in I if none of the following holds:*

1. *There is a single resident r_i and a hospital h_j , where r_i finds h_j acceptable, such that either r_i is unassigned in M or prefers h_j to $M(r_i)$, and either h_j is undersubscribed in M or prefers r_i to some member of $M(h_j)$.*
2. *There is couple (r_i, r_j) and a hospital h_k such that either*
 - (a) *(r_i, r_j) prefers $(h_k, M(r_j))$ to $(M(r_i), M(r_j))$, and either h_k is undersubscribed in M or prefers r_i to some member of $M(h_k) \setminus \{r_j\}$ or*
 - (b) *(r_i, r_j) prefers $(M(r_i), h_k)$ to $(M(r_i), M(r_j))$, and either h_k is undersubscribed in M or prefers r_j to some member of $M(h_k) \setminus \{r_i\}$.*
3. *There is a couple (r_i, r_j) and a pair of (not necessarily distinct) hospitals $h_k \neq M(r_i)$, $h_l \neq M(r_j)$ such that (r_i, r_j) finds (h_k, h_l) acceptable, and either (r_i, r_j) is unassigned or prefers the joint assignment (h_k, h_l) to $(M(r_i), M(r_j))$, and either*
 - (a) *$h_k \neq h_l$, and h_k (respectively h_l) is either undersubscribed in M or prefers r_i (respectively r_j) to at least one of its assignees in M ; or*
 - (b) *$h_k = h_l$, and h_k has two free posts in M , i.e., $c_k - |M(h_k)| \geq 2$; or*
 - (c) *$h_k = h_l$, and h_k has one free post in M , i.e., $c_k - |M(h_k)| = 1$, and h_k prefers at least one of r_i, r_j to some member of $M(h_k)$; or*
 - (d) *$h_k = h_l$, h_k is full in M , h_k prefers r_i to some $r_s \in M(h_k)$, and h_k prefers r_j to some $r_t \in M(h_k) \setminus \{r_s\}$.*

A resident and hospital, or a couple and hospital pair, satisfying one of the above conditions, is called a blocking pair of M and is said to block M .

Existing algorithmic results for HRC. An instance I of HRC need not admit a stable matching [34]. We call I *solvable* if it admits a stable matching, and *unsolvable* otherwise. Also an instance of HRC may admit stable matchings of differing sizes [2]. Further, the problem of deciding whether a stable matching exists in an instance of HRC is NP-complete, even in the restricted case where there are no single residents and each hospital has capacity 1 [28,33]. The decision problem is also W[1]-hard [25] when parameterized by the number of couples.

In many practical applications of HRC the residents' preference lists are short. Let (α, β, γ) -HRC denote the restriction of HRC in which each single resident's preference list contains at most α hospitals, each couple's preference list contains at most β pairs of hospitals and each hospital's preference list contains at most γ residents. Biró et al. [8] showed that deciding whether an instance of $(0, 2, 2)$ -HRC admits a stable matching is NP-complete.

Heuristics for HRC were described and compared experimentally by Biró et al. [5]. As far as exact algorithms are concerned, Biró et al. [8] gave an Integer Programming (IP) formulation for finding a maximum cardinality stable matching (or reporting that none exists) in an arbitrary instance of HRC and presented an empirical evaluation of an implementation of their model, showing that their formulation was capable of solving instances of the magnitude of those arising in the SFAS application. Further algorithmic results for HRC are given in [6, 23, 26].

Most-stable matchings. Given that a stable matching need not exist in a given HRC instance I , a natural question to ask is whether there is some other matching that might be the best alternative amongst the matchings in I . Roth [35, 36] argued that instability in the outcome of an allocation process gives participants a greater incentive to circumvent formal procedures; it follows minimising the amount of instability might be a desirable objective. Eriksson and Häggström [12] suggested that the number of blocking pairs admitted by a matching is a meaningful way to measure its degree of instability.

Define $bp(M)$ to be the set of blocking pairs relative to a matching M in I , and define a *most-stable matching* to be a matching M for which $|bp(M)|$ is minimum, taken over all matchings in I . Clearly if I admits a stable matching M , then M is a most-stable matching in I . Let MIN BP HRC denote the problem of finding a most-stable matching, given an instance of HRC. Most-stable matchings have been studied from an algorithmic point of view in various matching problem contexts [1, 9, 10, 13, 18, 19] (see [23] for more details), including in humanitarian organisations [38]. Define (α, β, γ) -MIN BP HRC to be the restriction of MIN BP HRC to instances of (α, β, γ) -HRC.

Contribution of this work. In Section 2 we show that $(\infty, 1, \infty)$ -MIN BP HRC is NP-hard and not approximable within $n_1^{1-\varepsilon}$, for any $\varepsilon > 0$, unless $P=NP$ (recall that n_1 is the number of residents in a given instance). In this highly restricted case of MIN BP HRC, each couple finds only one hospital pair acceptable and each hospital has capacity 1 (∞ refers to preference lists of unbounded length). We also show that $(\infty, \infty, 1)$ -MIN BP HRC and $(2, 1, 2)$ -MIN BP HRC are solvable in polynomial time. These results help to narrow down the search for the boundary

between polynomial-time solvable and NP-hard restrictions of MIN BP HRC (recall that $(0, 2, 2)$ -MIN BP HRC is NP-hard [8]).

In Section 3 we present the first IP model for MIN BP HRC; indeed this model can be used to find a most-stable matching of maximum cardinality. This formulation extends our earlier IP model for HRC, presented in [8]. Then in Section 4 we present data from an empirical evaluation of an implementation of the IP model for MIN BP HRC applied to randomly-generated instances. We measure the mean solution time, mean size of a most-stable matching and mean number of blocking pairs admitted by a most-stable matching when varying (i) the number of residents, (ii) the number of couples, (iii) the number of hospitals and (iv) the lengths of the residents' preference lists. Our main finding is that, over the 28,000 instances considered, the number of blocking pairs admitted by a most-stable matching is very small: it is usually at most 1, and never more than 2. This suggests that in a given HRC instance in practice, even if a stable matching does not exist, we may be able to find a matching with only a very small amount of instability.

Finally, in Section 5 we present the first Constraint Programming (CP) model for MIN BP HRC and evaluate its performance compared to the IP model over the instances used for the empirical analysis in Section 4. We observe that on average, the CP model is about 1.15 times faster than the IP model, and when presolving is applied to the CP model, it is on average 8.14 times faster.

Related work. Drummond et al. [11] presented SAT and IP encodings of HRC and investigated empirically their performance, along with two earlier heuristics for HRC, on randomly-generated instances. Their main aim was to measure the time taken to find a stable matching or report that none exists, and the proportion of solvable instances. They found that the SAT encoding gave the fastest method and was generally able to resolve the solvability question for the highest proportion of instances. In another paper [30], the same authors conducted further empirical investigations on random instances using an extension of their SAT encoding to determine how many stable matchings were admitted, and whether a resident Pareto optimal stable matching existed. We remark that the results in [11, 30] are not directly comparable to ours, because the stability definition considered in those papers is slightly weaker than that given by Definition 1. See Appendix A for a discussion of this issue.

Hinder [20] presented an IP model for a general stable matching problem with contracts, which includes HRC as defined here, as a special case. He conducted an empirical study on randomly-generated instances, comparing the performance of the IP model, its LP relaxation and a previously-published heuristic. Hinder showed that the LP relaxation finds stable matchings (when they exist) with much higher probability than the heuristic, and with probability quite close to the true value given by the IP model. The IP model terminates surprisingly quickly when the number of residents belonging to a couple is 10%, but it should be emphasised that in Hinder's random instances, all hospitals have capacity 1. In such a case our IP/CP models would be much simpler and need not involve the

constraints corresponding to stability criteria 3(b), 3(c) and 3(d) in Definition 1, thus our runtime results are not directly comparable to Hinder's.

To the best of our knowledge there have been no previous CP models for HRC, though a CP model for HR was given in [24], extending an earlier CP model for the classical Stable Marriage problem, the 1-1 restriction of HR [16]. A detailed survey of CP models for stable matching problems is given in [23, Section 2.5].

Nguyen and Vohra [29] proved a remarkable result, namely that it is always possible to find a stable matching in an instance of HRC if the capacity of each hospital can be adjusted (up or down) by at most 4, with the total capacity of the hospitals increasing by at most 9.

2 Complexity results for MIN BP HRC

In this section we present complexity and approximability results for MIN BP HRC in the case that preference lists of some or all of the agents are of bounded length. We begin with $(\infty, 1, \infty)$ -MIN BP HRC, the restriction in which each couple lists only one hospital pair on their preference list. Even in this highly restricted case, the problem of finding a most-stable matching is NP-hard and difficult to approximate. The proof of this result, given in Appendix B, begins by showing that, given an instance of $(\infty, 1, \infty)$ -HRC, the problem of deciding whether a stable matching exists is NP-complete. Then a gap-introducing reduction is given from this problem to $(\infty, 1, \infty)$ -MIN BP HRC.

Theorem 2 *$(\infty, 1, \infty)$ -MIN BP HRC is NP-hard and not approximable within a factor of $n_1^{1-\varepsilon}$, for any $\varepsilon > 0$, unless $P=NP$, where n_1 is the number of residents in a given instance. The result holds even if each hospital has capacity 1.*

We now turn to the case that hospitals' lists are of bounded length. It will be helpful to introduce the notion of a *fixed assignment* in a given HRC instance I . This involves either (i) a resident-hospital pair (r_i, h_j) such that h_j is the first choice of r_i , and r_i is among the first c_j choices of h_j , or (ii) a pair comprising a couple (r_i, r_j) and a pair of hospitals (h_p, h_q) such that h_p (resp. h_q) is the first choice of r_i (resp. r_j), and r_i (resp. r_j) is among the first c_p (resp. c_q) choices of h_p (resp. h_q). Clearly any stable matching must contain all the fixed assignments in I . By eliminating the fixed assignments iteratively, we arrive at the following straightforward result for $(\infty, \infty, 1)$ -HRC (the proofs of all the results stated in this section from this point onwards can be found in Appendix C).

Proposition 3 *An instance I of $(\infty, \infty, 1)$ -HRC admits exactly one stable matching, which can be found in polynomial time.*

We now consider the $(2, 1, 2)$ -HRC case. The process of *satisfying* a fixed assignment involves matching together the resident(s) and hospital(s) involved, deleting the agents themselves (and removing them from the remaining preference lists). This may uncover further fixed assignments, which themselves can be satisfied. Once this process terminates, we say that all fixed assignments have been *iteratively satisfied*. Let I be the $(2, 1, 2)$ -HRC instance that remains. It turns out that I has a special structure, as the following result indicates.

Lemma 4 *An arbitrary instance of $(2, 1, 2)$ -HRC involving at least one couple and in which all fixed assignments have been iteratively satisfied must be constructed from sub-instances of the form shown in Figure 8 (see Appendix C) in which all of the hospitals have capacity 1.*

It is then straightforward to find a most-stable matching in each such sub-instance.

Lemma 5 *Let I' be an instance of $(2, 1, 2)$ -HRC of the form shown in Figure 8 in Appendix C. If I' has an even number of couples then I' admits a stable matching M . Otherwise I' admits a matching M such that $|bp(M)| = 1$ in I' .*

Using Lemmas 4 and 5, it follows that we can find a most-stable matching in an instance I of $(2, 1, 2)$ -HRC as follows. Assume that M_0 is the matching in I in which all fixed assignments have been iteratively satisfied, and assume that the corresponding deletions have been made from the preference lists in I , yielding instance I' . Lemma 4 shows that I' is a union of disjoint sub-instances I_1, I_2, \dots, I_t , where each I_j is of the form shown in Figure 8 in Appendix C ($1 \leq j \leq t$). Let j ($1 \leq j \leq t$) be given and let N_j be the number of couples in I_j . Lemma 5 implies that, if N_j is even, we may find a stable matching M_j in I_j , otherwise we may find a matching M_j in I_j such that $|bp(M_j)| = 1$ in I_j . It follows that $M = \cup_{j=0}^t M_j$ is a most-stable matching in I . This leads to the following result.

Theorem 6 *$(2, 1, 2)$ -MIN BP HRC is solvable in polynomial time.*

It remains open to resolve the complexity of $(p, 1, q)$ -HRC for constant values of p and q where $\max\{p, q\} \geq 3$.

3 An Integer Programming formulation for MIN BP HRC

In this section we describe our IP model for MIN BP HRC, which extends the earlier IP model for HRC presented in [8] (we discuss relationships between the two models at the end of this section). Let I be an instance of HRC; we will denote by J the IP model corresponding to I . Due to space limitations we will only present some of the constraints in J ; the full description of J is contained in Appendix D.

Notation. We first define some required notation in I . Without loss of generality, suppose residents $r_1, r_2 \dots r_{2c}$ are in couples. Thus $r_{2c+1}, r_{2c+2} \dots r_{n_1}$ comprise the single residents. Again, without loss of generality, suppose that the couples are (r_{2i-1}, r_{2i}) ($1 \leq i \leq c$). Suppose that the joint preference list of a couple $\mathcal{C}_i = (r_{2i-1}, r_{2i})$ is $(h_{\alpha_1}, h_{\beta_1}), (h_{\alpha_2}, h_{\beta_2}) \dots (h_{\alpha_l}, h_{\beta_l})$. From this list we say that $h_{\alpha_1}, h_{\alpha_2} \dots h_{\alpha_l}$ and $h_{\beta_1}, h_{\beta_2} \dots h_{\beta_l}$ are the *individual* preference lists for r_{2i-1} and r_{2i} respectively. Let $l(r_i)$ denote the length of a resident r_i 's individual preference list (regardless of whether r_i is a single resident or r_i belongs to a couple).

For a resident $r_i \in R$ (whether single or a member of a couple), let $\text{pref}(r_i, p)$ denote the hospital at position p of r_i 's individual preference list. For an acceptable resident-hospital pair (r_i, h_j) , let $\text{rank}(h_j, r_i) = q$ denote the rank that hospital h_j assigns resident r_i , where $1 \leq q \leq l(h_j)$. Thus, $\text{rank}(h_j, r_i)$ is equal to the number of residents that h_j prefers to r_i plus 1.

Further, for each j ($1 \leq j \leq n_2$) and q ($1 \leq q \leq l(h_j)$), let the set $R(h_j, q)$ contain resident-position pairs (r_i, p) such that $r_i \in R$ is assigned a rank of q by h_j and h_j is in position p ($1 \leq p \leq l(r_i)$) on r_i 's individual list. Hence

$$R(h_j, q) = \{(r_i, p) \in R \times \mathbb{Z} : \text{rank}(h_j, r_i) = q \wedge 1 \leq p \leq l(r_i) \wedge \text{pref}(r_i, p) = h_j\}.$$

Variables in the IP model. For each i ($1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_i)$), J has a variable $x_{i,p} \in \{0, 1\}$ such that $x_{i,p} = 1$ if and only if r_i is assigned to his p^{th} -choice hospital. Also, for each i ($1 \leq i \leq n_1$) and $p = l(r_i) + 1$, J has a variable $x_{i,p} \in \{0, 1\}$ such that $x_{i,p} = 1$ if and only if r_i is unassigned. Let $X = \{x_{i,p} : 1 \leq i \leq n_1 \wedge 1 \leq p \leq l(r_i) + 1\}$.

J also contains variables $\theta_{i,p} \in \{0, 1\}$ for each i ($1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_i)$). The intuitive meaning of a variable $\theta_{i,p}$ is that $\theta_{i,p} = 1$ if and only if resident r_i is involved in a blocking pair with the hospital at position p on his individual preference list, either as a single resident or as part of a couple.

Constraints in the IP model. We firstly add constraints to J which force every variable to be binary valued. Next we ensure that matching constraints are satisfied, as follows. As each resident $r_i \in R$ is assigned to exactly one hospital or is unassigned (but not both), $\sum_{p=1}^{l(r_i)+1} x_{i,p} = 1$ must hold for all i ($1 \leq i \leq n_1$). Similarly, since a hospital h_j may be assigned at most c_j residents, $x_{i,p} = 1$ where $\text{pref}(r_i, p) = h_j$ for at most c_j residents, and hence for all j ($1 \leq j \leq n_2$), $\sum_{i=1}^{n_1} \sum_{p=1}^{l(r_i)} \{x_{i,p} \in X : \text{pref}(r_i, p) = h_j\} \leq c_j$ must hold.

For each couple (r_{2i-1}, r_{2i}) , r_{2i-1} is unassigned if and only if r_{2i} is unassigned, and r_{2i-1} is assigned to the hospital in position p in their individual list if and only if r_{2i} is assigned to the hospital in position p in their individual list. Thus for all i ($1 \leq i \leq c$) and p ($1 \leq p \leq l(r_{2i-1}) + 1$), $x_{2i-1,p} = x_{2i,p}$ must hold,

The remaining constraints in J allow the number of blocking pairs of a given matching to be counted. Each such constraint deals with a specific type of blocking pair that satisfies a given part of Definition 1. It allows a blocking pair to exist involving either (i) a single resident r_i with the hospital at some position p on his list, or (ii) a couple (r_{2i-1}, r_{2i}) with the hospital pair at some position p on their joint list, if and only if $\theta_{i,p} = 1$. We illustrate the construction of J by giving the constraint corresponding to so-called ‘‘Type 1’’ blocking pairs, involving involve single residents, where Condition 1 of Definition 1 is satisfied. The other constraints may be dealt with in a similar fashion – see Appendix D for further details.

Type 1 blocking pairs. In a matching M in I , if a single resident $r_i \in R$ is unassigned or has a worse partner than some hospital $h_j \in H$ where $\text{pref}(r_i, p) = h_j$ and $\text{rank}(h_j, r_i) = q$ then h_j must be fully subscribed with better partners than r_i , for otherwise (r_i, h_j) blocks M . Hence if r_i is unassigned or has worse

partner than h_j , i.e., $\sum_{p'=p+1}^{l(r_i)+1} x_{i,p'} = 1$, and h_j is not fully subscribed with better partners than r_i , i.e., $\sum_{q'=1}^{q-1} \{x_{i',p''} \in X : (r_{i'}, p'') \in R(h_j, q')\} < c_j$, then we require $\theta_{i,p} = 1$ to count this blocking pair. Thus, for each i ($2c+1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_i)$) we obtain the following constraint where $\text{pref}(r_i, p) = h_j$ and $\text{rank}(h_j, r_i) = q$:

$$c_j \left(\left(\sum_{p'=p+1}^{l(r_i)+1} x_{i,p'} \right) - \theta_{i,p} \right) \leq \sum_{q'=1}^{q-1} \{x_{i',p''} \in X : (r_{i'}, p'') \in R(h_j, q')\}.$$

Objective functions in the IP model. A maximum cardinality most-stable matching M is a matching of maximum cardinality, taken over all most-stable matchings in I . To compute a maximum most-stable matching in J , we apply two objective functions in sequence.

First we find an optimal solution in J that minimises the number of blocking pairs. To this end we apply the objective function $\min \sum_{i=1}^{n_1} \sum_{p=1}^{l(r_i)} \theta_{i,p}$.

The matching M corresponding to an optimal solution in J will be a most-stable matching in I . Let $k = |bp(M)|$. Now we seek a maximum cardinality matching in I with at most k blocking pairs. Thus we add the following constraint to J , which ensures that, when maximising on cardinality, any solution also has at most k blocking pairs: $\sum_{i=1}^{n_1} \sum_{p=1}^{l(r_i)} \theta_{i,p} \leq k$.

The final step is to maximise the size of the matching, subject to the matching being most-stable. This involves optimising for a second time, this time using the following objective function: $\max \sum_{i=1}^{n_1} \sum_{p=1}^{l(r_i)} x_{i,p}$.

The following result, which establishes the correctness of the IP formulation, is proved in Appendix D.

Theorem 7 *Given an instance I of MIN BP HRC, let J be the corresponding IP model as defined above. A maximum cardinality most-stable matching in I is exactly equivalent to an optimal solution to J .*

We remark that the IP model presented in this section develops the earlier model for HRC [8] with the addition of the $\theta_{i,p}$ variables. There are similarities between the constraints (with these variables omitted) when comparing the two models. However in the HRC model [8] essentially all stability constraints had to be satisfied, whereas in the MIN BP HRC model a blocking pair is allowed at the expense of a $\theta_{i,p}$ variable having value 1, which allows the number of blocking pairs to be counted. Suitable placement of the $\theta_{i,p}$ variables within the constraints from the HRC model allows this condition on the $\theta_{i,p}$ variables to be enforced.

4 Empirical results from the IP model for MIN BP HRC

In this section we present data from an empirical evaluation of an implementation of the IP model for finding a maximum cardinality most-stable matching in an instance of MIN BP HRC. We considered the following properties for randomly-generated HRC instances: the time taken to find a maximum cardinality most-stable matching, the size of a maximum cardinality most-stable matching and the number of blocking pairs admitted by a most-stable matching. We show how these properties varied as we modified the number of residents, the percentage of residents involved in couples, the number of hospitals and the lengths of residents' preference lists in the constructed instances.

Methodology. We ran all the experiments on an implementation of the IP model using the CPLEX 12.4 Java Concert API applied to randomly-generated instances of HRC¹. In these instances, the preference lists of residents and hospitals were constructed to take into account of the fact that, in reality, some hospitals and residents are more popular than others, respectively. Typically, the most popular hospital in the SFAS context had 5-6 times as many applicants as the least popular, and the numbers of applicants to the other hospitals were fairly uniformly distributed between the two extremes. Our constructed instances reflected this real-world behaviour. For more details about the construction of the instances and the correctness testing methodology, the reader is referred to [26, Chapters 6,7].

All experiments were carried out on a desktop PC with an Intel i5-2400 3.1Ghz processor with 8Gb of memory running Windows 7. To find a most-stable matching in an instance I of HRC we applied the following procedure. We first used the HRC IP implementation presented in [8] to find a maximum cardinality stable matching M in I if one exists. Clearly, if I is solvable then M is a maximum cardinality most-stable matching. However, if I was found to be unsolvable, we applied the MIN BP HRC IP model to I . In this case we applied a lower bound of 1 to the number of blocking pairs in a most-stable matching in I since we knew that no stable matching existed. All instances were allowed to run to completion. We remark that the MIN BP HRC model appears to be much more difficult to solve than the HRC model presented in [8], and thus the largest instances sizes considered here are smaller than the largest ones generated in the experimental evaluation in [8].

Experiment 1. In the first experiment we increased the number of residents while maintaining a constant ratio of couples, hospitals and posts to residents. For various values of x ($50 \leq x \leq 150$) in increments of 20, 1000 randomly generated instances were created containing x residents, $0.1x$ couples (and hence $0.8x$ single residents) and $0.1x$ hospitals with x available posts that were randomly distributed amongst the hospitals. Each resident's preference list contained a minimum of 3 and a maximum of 5 hospitals. Figure 1 (and indeed all the figures in this section) shows the mean time taken to find a maximum cardinality

¹ All generated instances can be obtained from <http://dx.doi.org/10.5525/gla.researchdata.303>.

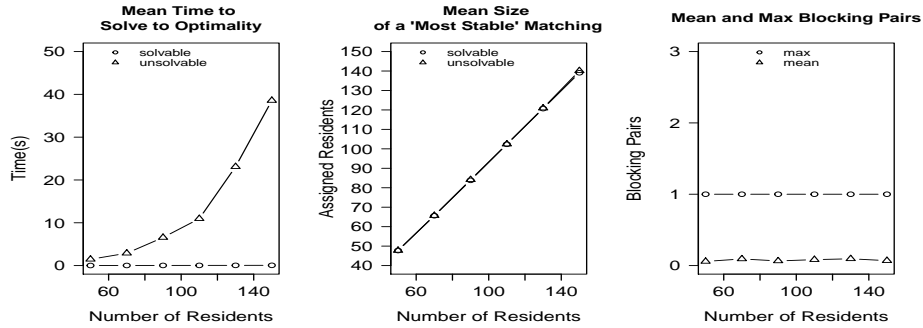


Fig. 1. Empirical results for Experiment 1.

most-stable matching, the mean size of a maximum cardinality most-stable solution (in each case over both solvable and unsolvable instances), and the mean and maximum number of blocking pairs admitted by most-stable matchings.

The results show that the time taken to find an optimal solution increases with x , with the MIN BP HRC formulation being more difficult to solve in general than the HRC formulation. The mean size of an optimal solution increases with x for both solvable and unsolvable instances (it is around 95% of x for $x = 50$, decreasing to around 93% of x for $x = 150$, with the optimal matching size for unsolvable instances being very slightly larger than that for solvable instances). Perhaps most interestingly, the maximum number of blocking pairs was 1, with the mean at most 0.1, and the mean number of unsolvable instances being 77.

Experiment 2. In our second experiment we increased the percentage of residents involved in couples while maintaining the same numbers of residents, hospitals and posts. For various values of x ($0 \leq x \leq 30$) in increments of 5, 1000 randomly generated instances were created containing 100 residents, x couples (and hence $100 - 2x$ single residents) and 10 hospitals with 100 available posts that were unevenly distributed amongst the hospitals. Each resident's preference list contained a minimum of 3 and a maximum of 5 hospitals. The results for all values of x are displayed in Figure 2.

The results show that the time taken to find an optimal solution increases with x ; again the MIN BP HRC formulation is more difficult to solve in general than the HRC formulation. The mean size of an optimal solution decreases with x for both solvable and unsolvable instances; again the optimal matching size for unsolvable instances is slightly larger than that for solvable instances. As for Experiment 1, the maximum number of blocking pairs was 1, with the number of unsolvable instances increasing from 50 for $x = 5$ to 224 for $x = 30$.

Experiment 3. In our third experiment we increased the number of hospitals in the instance while maintaining the same numbers of residents, couples and posts. For various values of x ($10 \leq x \leq 100$) in increments of 10, 1000 randomly generated instances were created containing 100 residents, 10 couples (and hence

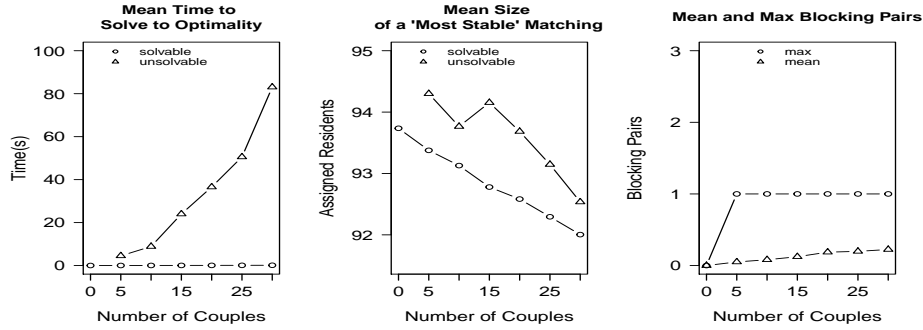


Fig. 2. Empirical results for Experiment 2.

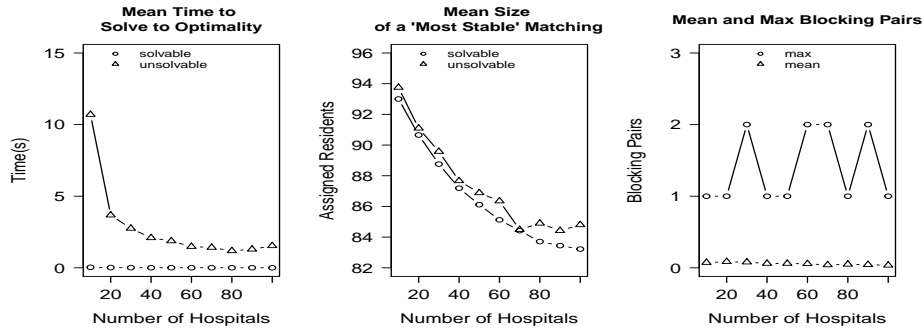


Fig. 3. Empirical results for Experiment 3.

80 single residents) and x hospitals with 100 available posts that were unevenly distributed amongst the hospitals. Each resident's preference list contained a minimum of 3 and a maximum of 5 hospitals. The results for all values of x are displayed in Figure 3.

The results show that the time taken to find an optimal solution decreases with x ; again the MIN BP HRC model solution time is slower than that for the HRC model. Clearly the problem is becoming less constrained as the number of hospitals increases. Also the mean size of an optimal solution decreases with x for both solvable and unsolvable instances; again the optimal matching size for unsolvable instances is slightly larger than that for solvable instances. This time the maximum number of blocking pairs was 2, with the mean number of blocking pairs decreasing from 0.08 for $x = 20$ to 0.04 for $x = 100$.

Experiment 4. In our last experiment, we increased the length of the individual preference lists for the residents in the instance while maintaining the same numbers of residents, couples, hospitals and posts. For various values of x ($2 \leq x \leq 6$), 1000 randomly generated instances were created containing 100 residents, 10 couples (and hence 80 single residents) and 10 hospitals with 100

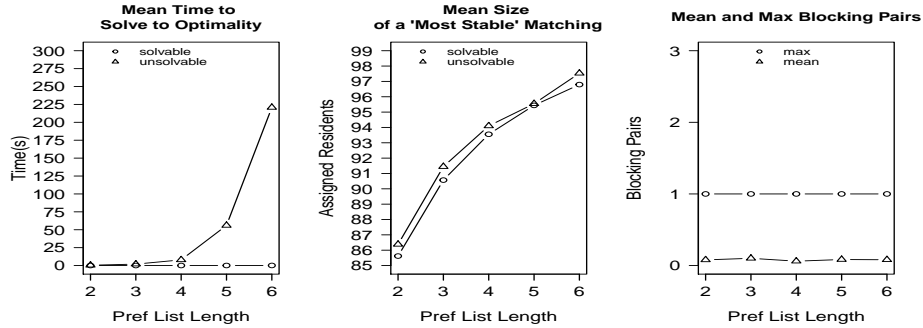


Fig. 4. Empirical results for Experiment 4.

available posts that were unevenly distributed amongst the hospitals. Each resident's preference list contained exactly x hospitals. The results for all values of x are displayed in Figure 4.

The results show that increasing the preference list length makes the problem harder to solve; again the MIN BP HRC model is slower to solve than the HRC model. Also the mean size of an optimal solution increases with x for both solvable and unsolvable instances as more options become available in the preference lists (from 86.4 for $x = 2$ to 97.5 for $x = 6$ in the case of unsolvable instances); again the optimal matching size for unsolvable instances is slightly larger than that for solvable instances. The maximum number of blocking pairs was 1, with the mean at most 0.1, and the mean number of unsolvable instances being 81.

Discussion. The results presented in this section suggest that, even as we increase the number of residents or hospitals, the percentage of residents involved in a couple or the length of the residents' preference lists, the number of blocking pairs admitted by a most-stable matching is very low. For most of the 28,000 instances generated in our experimental evaluation, the most-stable matchings found admitted at most 1 blocking pair, and the maximum number of blocking pairs admitted by any most-stable matching was never more than 2. These findings are essentially consistent with the results of Nguyen and Vohra [29], who showed that an unsolvable HRC instance only requires a small amount of perturbation in order to become solvable. Further empirical investigation is required to determine whether this behaviour is replicated for larger HRC instance sizes.

5 A Constraint Programming model for MIN BP HRC

In addition to the IP model, we designed a Constraint Programming model for MIN BP HRC and implemented this using the MiniZinc constraint modelling language.

We assume that residents' preference lists are given by integer variables $rpref[i][j]$, which play a similar role to $pref(r_i, j)$ in the IP model, and that

hospitals' ranking arrays are given by integer variables $h\text{rank}[h, i]$, which are analogous to $\text{rank}(h_j, r_i)$ in the IP model. The lengths of the preference lists of a resident r_i and a hospital h_j are given by $r\text{pref_len}[i]$ and $h\text{pref_len}[j]$ respectively. The capacity of a hospital h_j is given by $h\text{osp_cap}[j]$.

For each single resident r_i , the model includes an integer variable $\text{single_pos}[i]$ with domain $(1, \dots, l(r_i) + 1)$, where $l(r_i)$ is the value of $r\text{pref_len}[i]$, which takes the value j if r_i is assigned her j th-choice hospital, or $l(r_i) + 1$ if r_i is unassigned. For each couple i , we include an integer variable $\text{coup_pos}[i]$ with a similar interpretation.

Each single resident's $\text{single_pos}[i]$ variable is channelled to an array of $l(r_i)$ boolean variables $\text{single_assigned}[i]$, such that $\text{single_assigned}[i][j] = \text{true}$ if and only if $\text{single_pos}[i] = j$, and a variable $\text{single_unassigned}[i]$, such that $\text{single_unassigned}[i] = \text{true}$ if and only if $\text{single_pos}[i] = l(r_i) + 1$. Similarly, we have boolean coup_assigned and coup_unassigned variables for each couple.

For each hospital i , and each position j on hospital i 's preference list, we have a boolean variable $\text{hosp_assigned}[i][j]$ which is true if and only if hospital i is assigned its j th-choice resident. We include a constraint to ensure that $\text{hosp_assigned}[i][j] = \text{true}$ if and only if a corresponding single_assigned or coup_assigned variable is also true . Furthermore, each hospital has a linear inequality constraint to ensure that its capacity is not exceeded.

For each position on the preference list of a single resident or couple, we create a boolean variable $\text{single_bp}[i][j]$ or $\text{coup_bp}[i][j]$ indicating whether the resident or couple, along with their j th-choice hospital, constitutes a blocking pair. For each type of blocking pair, we define a set of constraints and then give some brief intuition.

Type 1 blocking pairs

```
constraint forall (i in Singles) (
  forall(j in 1 ... rpref_len[i]) (
    let {int: h = rpref[i, j], int: q = hrank[h, i]} in
      single_pos[i] > j ∧ hosp_would_prefer(h, q) ⇒ single_bp[i, j]);
```

The *hosp_would_prefer* predicate for a hospital h and a position q on the preference list of h takes the value true if and only if h has fewer than $\text{hosp_cap}[h]$ assigned residents in positions strictly preferable to position q on its preference list. (Note the redundancy in this predicate: all we actually need is the first $\text{sum}(\dots)(\dots) < \text{hosp_cap}[h]$ constraint; the $\text{sum}(\dots)(\dots) > 0$ constraint improves propagation.)

```
predicate hosp_would_prefer(int:h, int:q) =
  if q ≤ hosp_cap[h] then
    true
  else
    sum(k in 1 ... q - 1)(bool2int(hosp_assigned[h, k])) < hosp_cap[h] ∨
    sum(k in q + 1 ... hpref_len[h])(bool2int(hosp_assigned[h, k])) > 0
  endif;
```

The constraint for Type 1 blocking pairs thus sets $single_bp[i, j]$ to **true** if and only if r_i is unassigned or prefers h to his partner, and h is undersubscribed or prefers r_i at least one of its assignees, where $h = rpref[i, j]$.

Type 2a/b blocking pairs

```
constraint forall (i in Couples) (
  let {int: r1 = first_in_couple(i), int: r2 = second_in_couple(i)} in
  forall(j in 1 .. rpref_len[r1]) (
    let {int: h1 = rpref[r1, j], int: h2 = rpref[r2, j],
        int: q1 = hrank[h1, r1], int: q2 = hrank[h2, r2]} in
    coup_pos[i] > j  $\wedge$ 
    ((hosp_would_prefer_exc_partner(h1, h2, q1, q2)  $\wedge$ 
      h2 = rpref[r2, coup_pos[i]])  $\vee$ 
     (hosp_would_prefer_exc_partner(h2, h1, q2, q1)  $\wedge$ 
      h1 = rpref[r1, coup_pos[i]]))
     $\Rightarrow$  coup_bp[i, j]
  )
);
```

The *hosp_would_prefer_exc_partner* predicate on inputs $h1, h2, q1, q2$ (where $h1, h2$ are hospitals and $q1, q2$ are positions on their preference lists respectively) takes the value **true** if and only if (a) $h1 = h2, q1 < q2$ and the number of $h1$'s assignees that it prefers to its $q1$ th choice is less than $hosp_cap[h1] - 1$, or (b) $h1 \neq h2$ or $q1 > q2$ and the number of $h1$'s assignees that it prefers to its $q1$ th choice is less than $hosp_cap[h1]$.

```
predicate hosp_would_prefer_exc_partner(int:h1, int:h2, int:q1, int:q2) =
  if h1 = h2  $\wedge$  q1 < q2 then
    sum(k in 1 .. q1 - 1)(bool2int(hosp_assigned[h1, k])) < hosp_cap[h1] - 1
  else
    sum(k in 1 .. q1 - 1)(bool2int(hosp_assigned[h1, k])) < hosp_cap[h1]
  endif;
```

The constraint for Type 2a/b blocking pairs thus sets $coup_bp[i, j]$ to **true** if and only if couple $(r1, r2)$ prefer hospital pair $(h1, h2)$ to their joint assignment $(h3, h4)$, where *either*

- (a) $h2 = h4$ and either $h1$ is undersubscribed or prefers $r1$ to at least one assignee that is not $r2$ (if $r2$ is assigned to $h1$) *or*
- (b) $h1 = h3$ and either $h2$ is undersubscribed or prefers $r2$ to at least one assignee that is not $r1$ (if $r1$ is assigned to $h2$).

Type 3a blocking pairs

```
constraint forall (i in Couples) (
  let {int: r1 = first_in_couple(i), int: r2 = second_in_couple(i)} in
  forall(j in 1 .. rpref_len[r1]) where rpref[r1, j] != rpref[r2, j] (
```

```

    let {int:  $h1 = rpref[r1, j]$ , int:  $h2 = rpref[r2, j]$ ,
        int:  $q1 = hrank[h1, r1]$ , int:  $q2 = hrank[h2, r2]$ } in
     $hosp\_would\_prefer(h1, q1) \wedge hosp\_would\_prefer(h2, q2) \wedge$ 
     $h1 \neq rpref[r1, coup\_pos[i]] \wedge h2 \neq rpref[r2, coup\_pos[i]] \wedge$ 
     $coup\_pos[i] > j \Rightarrow coup\_bp[i, j]$ 
  )
);

```

The constraint for Type 3a blocking pairs thus sets $coup_bp[i, j]$ to **true** if and only if couple $(r1, r2)$ are unassigned or prefer $(h1, h2)$ to their joint assignment, whilst for each $k \in \{1, 2\}$, hk is undersubscribed or prefers rk to at least one of its assignees, where $(r1, r2)$ is the i th couple and $(h1, h2)$ is the hospital pair at position j of their joint list.

Type 3b/c/d blocking pairs

```

constraint forall ( $i$  in  $Couples$ ) (
  let {int:  $r1 = first\_in\_couple(i)$ , int:  $r2 = second\_in\_couple(i)$ } in
  forall( $j$  in  $1 \dots rpref\_len[r1]$  where  $rpref[r1, j] = rpref[r2, j]$ ) (
    let {int:  $h = rpref[r1, j]$ , int:  $q1 = hrank[h, r1]$ ,
        int:  $q2 = hrank[h, r2]$ } in
    if  $q1 < q2$  then
       $hosp\_would\_prefer2(h, q1) \wedge hosp\_would\_prefer(h, q2)$ 
    else
       $hosp\_would\_prefer(h, q1) \wedge hosp\_would\_prefer2(h, q2)$ 
    end if  $\wedge$ 
     $coup\_pos[i] > j \wedge$ 
     $h \neq rpref[r1, coup\_pos[i]] \wedge h \neq rpref[r2, coup\_pos[i]]$ 
     $\Rightarrow coup\_bp[i, j]$ 
  )
);

```

The $hosp_would_prefer2$ predicate for a hospital h and a position q on the preference list of h takes the value **true** if and only if h has fewer than $hosp_cap[h] - 1$ assigned residents in positions strictly preferable to position q on its preference list. (Note the redundancy in this predicate: all we actually need is the first $sum(\dots)(\dots) < hosp_cap[h] - 1$ constraint; the $sum(\dots)(\dots) > 1$ constraint improves propagation.)

```

predicate  $hosp\_would\_prefer2(\mathbf{int}:h, \mathbf{int}:q) =$ 
   $\sum(k \text{ in } 1 \dots q - 1)(\mathbf{bool2int}(hosp\_assigned[h, k])) < hosp\_cap[h] - 1 \vee$ 
   $\sum(k \text{ in } q + 1 \dots hpref\_len[h])(\mathbf{bool2int}(hosp\_assigned[h, k])) > 1 ;$ 

```

The constraint for Type 3b/c/d blocking pairs thus sets $coup_bp[i, j]$ to **true** if and only if couple $(r1, r2)$ are unassigned or prefer (h, h) to their joint assignment, whilst h either has two free posts (Type 3b), or h has one free post and prefers one of $r1$ or $r2$ to at least one of its assignees (Type 3c), or h is full and and prefers $r1$ to some assignee rk , and prefers $r2$ to at least one of its

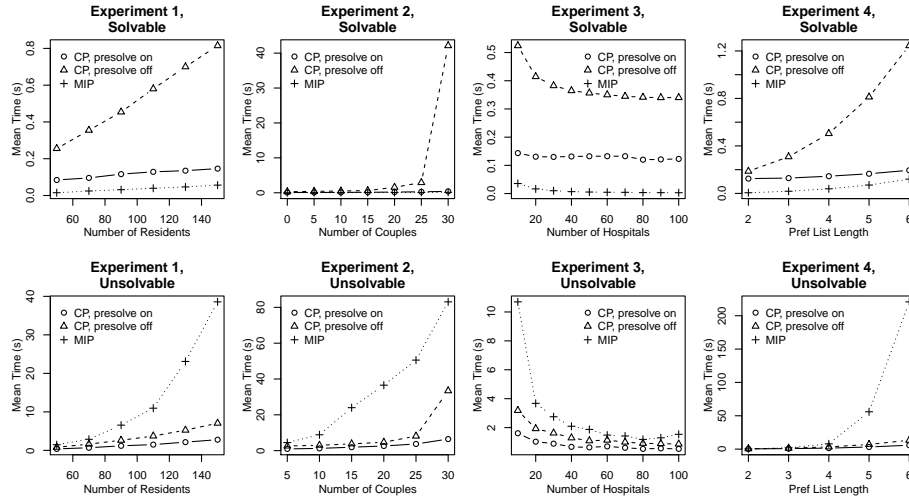


Fig. 5. Comparison of run times using CP (with and without presolve) and MIP models

assignees apart from rk (Type 3d), where $(r1, r2)$ is the i th couple and (h, h) is the hospital pair at position j of their joint list.

Experiments. The CP model was solved using the lazy clause solver Chuffed [44] on the same machine that was used for the experiments on the IP model as reported in Section 4. All instances were allowed to run to completion. We present results on the runtime of the CP model both with and without presolving. The presolve step, when included, specifies in advance which set S of resident-hospital pairs will block the solution (in practice we try out values of $k = 0, 1, 2, \dots$ and generate all subsets S of size k until we reach a feasible solution) and then performs preference list deletions in the knowledge that the pairs in S will block. This allows large reductions in the model size, and works well because the number of blocking pairs admitted by a most-stable matching is generally very small, as we saw in Section 4. We did not use presolve with the IP model, but we note that it may be possible to solve the IP model more quickly by carrying out this step.

Figure 5 plots the mean run times for each of the four experiments for the IP model and for the CP models with and without presolving: each plot in the top row shows results for the solvable instances in one experiment, and each plot in the bottom row shows corresponding results for the unsolvable instances. Table 1 shows the actual mean and median runtimes for each model, taken over all 28,000 instances \mathcal{I} across all four experiments, those instances from \mathcal{I} that were solvable and those from \mathcal{I} that were unsolvable.

The CP model without presolve generally performs unfavourably for solvable instances. Here, the IP model is faster than the CP model with presolve; this is likely to be due to the fact that for such instances, the earlier IP model for

Instance type	Mean			Median		
	IP model	CP model		IP model	CP model	
		No presolve	Presolve		No presolve	Presolve
All	2.568	2.237	0.315	0.031	0.430	0.129
Solvable	0.034	1.839	0.143	0.016	0.402	0.127
Unsolvable	30.781	6.669	1.276	8.948	3.240	1.395

Table 1. Summary of mean and median runtimes over all experiments (all timings are in seconds)

HRC [8] is used instead of the more complex IP formulation for MIN BP HRC. For unsolvable instances, the CP model (with or without presolve) is faster than the IP model. This is likely to be due to the fact that the CP model for MIN BP HRC is more compact than its IP counterpart, involving fewer variables and constraints. Comparing total run time summed across all 28,000 instances, the CP model was 1.15 times faster than the CP model without presolve, and the CP model with presolve was 8.14 times faster than the IP model.

When solving the CP model, the distribution of runtimes for the case without presolve had a very long right tail; 14 of the 28,000 instances accounted for over half of the total run time. The longest-running instance took 17,617 seconds, and surprisingly this was a solvable instance (generated for Experiment 2). For this reason, Table 1 shows median run times as well as mean run times; from this we can see that the median runtime for the IP model is lower than that for the CP models for all instances and for solvable instances. However for unsolvable instances, the median runtime for CP without presolve is 2.762 times faster than the median runtime for IP, and this factor increases to 6.414 for CP with presolve.

6 Concluding remarks

In this paper we have presented complexity and approximability results for MIN BP HRC, showing that the problem is NP-hard and very difficult to approximate even in highly restricted cases. We have then presented IP and CP models, together with empirical analyses of both models applied to randomly-generated HRC instances. Our main finding is that most-stable matchings admit a very small number of blocking pairs (in most cases at most 1, but never more than 2) on the instances we generated. We also showed that on average the CP model is faster than the IP model, with the performance of the CP model being enhanced if presolving was carried out. As far as future work is concerned, it would be interesting to determine the effect of presolving on the IP model, and more generally, to investigate further methods to enable the models to be scaled up to larger instances, such as column generation in the case of the IP model, and variable / value ordering heuristics in the case of the CP model.

Acknowledgements

We would like to thank anonymous reviewers of an earlier version of this paper for their valuable comments, including the suggestion of references [31, 37–39], which have helped to improve the presentation of this paper.

References

1. D.J. Abraham, P. Biró, and D.F. Manlove. “Almost stable” matchings in the Roommates problem. In *Proceedings of WAOA '05: the 3rd Workshop on Approximation and Online Algorithms*, volume 3879 of *Lecture Notes in Computer Science*, pages 1–14. Springer, 2006.
2. B. Aldershof and O.M. Carducci. Stable matching with couples. *Discrete Applied Mathematics*, 68:203–207, 1996.
3. M. Balinski and T. Sönmez. A tale of two mechanisms: student placement. *Journal of Economic Theory*, 84(1):73–94, 1999.
4. P. Biró. Student admissions in Hungary as Gale and Shapley envisaged. Technical Report TR-2008-291, University of Glasgow, Department of Computing Science, 2008.
5. P. Biró, R.W. Irving, and I. Schlotter. Stable matching with couples: an empirical study. *ACM Journal of Experimental Algorithmics*, 16, 2011. Section 1, article 2.
6. P. Biró and F. Klijn. Matching with couples: a multidisciplinary survey. *International Game Theory Review*, 15(2), 2013. article number 1340008.
7. P. Biró, D.F. Manlove, and I. McBride. The Hospitals / Residents problem with Couples: Complexity and integer programming models. Technical Report arXiv:1308.4534, Computing Research Repository, Cornell University Library, 2013. Available from <http://arxiv.org/abs/1308.4534>.
8. P. Biró, D.F. Manlove, and I. McBride. The Hospitals / Residents problem with Couples: Complexity and Integer Programming Models. In *Proceedings of SEA '14: the 8th Symposium on Experimental Algorithms*, volume 8504 of *Lecture Notes in Computer Science*, pages 10–21. Springer, 2014.
9. P. Biró, D.F. Manlove, and E.J. McDermid. “Almost-stable” matchings in the Roommates problem with bounded preference lists. *Theoretical Computer Science*, 432:10–20, 2012.
10. P. Biró, D.F. Manlove, and S. Mittal. Size versus stability in the marriage problem. *Theoretical Computer Science*, 411:1828–1841, 2010.
11. J. Drummond, A. Perrault, and F. Bacchus. SAT is an effective and complete method for solving stable matching problems with couples. In *Proceedings of IJCAI '15: the Twenty-Fourth International Joint Conference on Artificial Intelligence*, pages 518–525. AAAI Press, 2015.
12. K. Eriksson and O. Häggström. Instability of matchings in decentralized markets with various preference structures. *International Journal of Game Theory*, 36(3-4):409–420, 2008.
13. P. Floréen, P. Kaski, V. Polishchuk, and J. Suomela. Almost stable matchings by truncating the Gale-Shapley algorithm. *Algorithmica*, 58(1):102–118, 2010.
14. D. Gale and L.S. Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69:9–15, 1962.
15. M.R. Garey, D.S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1:237–267, 1976.

16. I.P. Gent, R.W. Irving, D.F. Manlove, P. Prosser, and B.M. Smith. A constraint programming approach to the stable marriage problem. In *Proceedings of CP '01: the 7th International Conference on Principles and Practice of Constraint Programming*, volume 2239 of *Lecture Notes in Computer Science*, pages 225–239. Springer, 2001.
17. D. Gusfield and R.W. Irving. *The Stable Marriage Problem: Structure and Algorithms*. MIT Press, 1989.
18. K. Hamada, K. Iwama, and S. Miyazaki. An improved approximation lower bound for finding almost stable stable maximum matchings. *Information Processing Letters*, 109(18):1036–1040, 2009.
19. K. Hamada, K. Iwama, and S. Miyazaki. The hospitals/residents problem with lower quotas. *Algorithmica*, 74(1):440–465, 2016.
20. O. Hinder. The stable matching linear program and an approximate rural hospital theorem with couples. In *Proceedings of WINE '15: the 11th Conference on Web and Internet Economics*, volume 9470 of *Lecture Notes in Computer Science*, page 433. Springer, 2015. Full version available from <http://stanford.edu/~ohinder/stability-and-lp/working-paper.pdf>.
21. R.W. Irving. Matching medical students to pairs of hospitals: a new variation on a well-known theme. In *Proceedings of ESA '98: the 6th Annual European Symposium on Algorithms*, volume 1461 of *Lecture Notes in Computer Science*, pages 381–392. Springer, 1998.
22. D. Maier and J.A. Storer. A note on the complexity of the superstring problem. Technical Report 233, Princeton University, Department of Electrical Engineering and Computer Science, Princeton, NJ, October 1977.
23. D.F. Manlove. *Algorithmics of Matching Under Preferences*. World Scientific, 2013.
24. D.F. Manlove, G. O'Malley, P. Prosser, and C. Unsworth. A Constraint Programming Approach to the Hospitals / Residents Problem. In *Proceedings of CP-AI-OR 2007: the 4th International Conference on Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization*, volume 4510 of *Lecture Notes in Computer Science*, pages 155–170. Springer, 2007.
25. D. Marx and I. Schlotter. Stable assignment with couples: parameterized complexity and local search. *Discrete Optimization*, 8:25–40, 2011.
26. I. McBride. *Complexity and Integer Programming Models for Generalisations of the Hospitals / Residents Problem*. PhD thesis, School of Computing Science, University of Glasgow, 2015.
27. E.J. McDermid and D.F. Manlove. Keeping partners together: Algorithmic results for the hospitals / residents problem with couples. *Journal of Combinatorial Optimization*, 19(3):279–303, 2010.
28. C. Ng and D.S. Hirschberg. Complexity of the stable marriage and stable roommate problem in three dimensions. Technical Report UCI-ICS 88-28, Department of Information and Computing Science, University of California, Irvine, 1988.
29. T. Nguyen and R. Vohra. Near feasible stable matchings. In *Proceedings of EC '15: the Sixteenth ACM Conference on Economics and Computation*, pages 41–42. ACM, 2015.
30. A. Perrault, J. Drummond, and F. Bacchus. Exploring strategy-proofness, uniqueness, and Pareto-optimality for the stable matching problem with couples. Technical Report arxiv:1505.03463, Computing Research Repository, Cornell University Library, 2015. Available from <http://arxiv.org/abs/1505.03463>.
31. P.A. Robards. Applying two-sided matching processes to the united states navy enlisted assignment process. Master's thesis, Naval Postgraduate School, Monterey, CA, 2001.

32. A. Romero-Medina. Implementation of stable solutions in a restricted matching market. *Review of Economic Design*, 3(2):137–147, 1998.
33. E. Ronn. NP-complete stable matching problems. *J. Algorithms*, 11:285–304, 1990.
34. A.E. Roth. The evolution of the labor market for medical interns and residents: a case study in game theory. *Journal of Political Economy*, 92(6):991–1016, 1984.
35. A.E. Roth. New physicians: A natural experiment in market organization. *Science*, 250:1524–1528, 1990.
36. A.E. Roth. A natural experiment in the organization of entry level labor markets: Regional markets for new physicians and surgeons in the UK. *American Economic Review*, 81:415–440, 1991.
37. M.M. Short. Analysis of the current navy enlisted detailing process. Master's thesis, Naval Postgraduate School, Monterey, CA, 2000.
38. M. Soldner. *Optimization and measurement in humanitarian operations: Addressing practical needs*. PhD thesis, Georgia Institute of Technology, 2014.
39. W. Yang, J.A. Giampapa, and K. Sycara. Two-sided matching for the U.S. Navy Detailing Process with market complication. Technical Report CMU-RI-TR-03-49, Robotics Institute, Carnegie-Mellon University, 2003.
40. Canadian Resident Matching Service website. <http://www.carms.ca>.
41. Central Applications Office Ireland website. <http://www.cao.ie>.
42. Japan Resident Matching Program website. <http://www.jrmp.jp>.
43. Matching in Practice website. <http://www.matching-in-practice.eu/>.
44. Higher Education Allocation in Ireland - Matching in Practice Website. <http://www.matching-in-practice.eu/higher-education-in-ireland>.
45. National Resident Matching Program website. <http://www.nrmp.org>.

Appendix

A Comparison of stability definitions

In this section we compare our stability definition for HRC given by Definition 1 with the definition adopted by Drummond et al. [11]. Suppose that I is an instance of HRC and M is a matching in I . Let $R' \subseteq R$. For a given hospital h_j , Drummond et al. defined $Ch_j(R')$ to be the set of residents that h_j would select from R' . That is, $Ch_j(R')$ is the maximal subset of R' such that, for all $r_i \in Ch_j(R')$, h_j finds r_i acceptable, h_j prefers r_i to any $r_k \in R' \setminus Ch_j(R')$ and $|Ch_j(R')| \leq c_j$. Then Drummond et al. defined the predicate $\text{willAccept}(h_j, R', M)$ to be true if and only if $R' \subseteq Ch_j(M(h_j) \cup R')$.

Now suppose that (r_i, r_j) is a couple in I who prefer the hospital pair (h_k, h_k) to their assigned hospital pair $(M(r_i), M(r_j))$, where h_k is full in M . According to Condition 3(b) of the stability definition of Drummond et al. [11], h_k will participate in a blocking pair with (r_i, r_j) if and only if $\text{willAccept}(h_k, \{r_i, r_j\}, M)$. According to Condition 3(d) of Definition 1, h_k will participate in a blocking pair with (r_i, r_j) if and only if h_k prefers r_i to some $r_s \in M(h_k)$, and h_k prefers r_j to some $r_t \in M(h_k) \setminus \{r_s\}$. Our Condition 3(d) is thus weaker than Condition 3(b) of Drummond et al. [11], meaning that our stability definition is stricter.

To illustrate the difference, consider the HRC instance I shown in Figure 6. Here h_1 has capacity 2, whilst each of h_2 and h_3 has capacity 1.

Residents		Hospitals			
$(r_1, r_2) : (h_1, h_1)$	(h_2, h_3)	$h_1 : r_1$	r_3	r_2	r_4
$r_3 : h_1$		$h_2 : r_1$			
$r_4 : h_1$		$h_3 : r_1$			

Fig. 6. An instance of HRC.

Let M be the matching $\{(r_1, h_2), (r_2, h_3), (r_3, h_1), (r_4, h_1)\}$. Then (r_1, r_2) forms a blocking pair of M with the hospital pair (h_1, h_1) according to the stability definition given in Definition 1, but this does not happen with respect to the stability definition of Drummond et al. [11]. In the latter case, $Ch_j(M(h_1) \cup \{r_1, r_2\}) = Ch_j(\{r_1, r_2, r_3, r_4\}) = \{r_1, r_3\} \not\supseteq \{r_1, r_2\}$. We would argue that (r_1, r_2) should form a blocking pair with (h_1, h_1) , because h_1 unequivocally improves by rejecting $\{r_3, r_4\}$ and taking on $\{r_1, r_2\}$ instead.

B Inapproximability result for $(\infty, 1, \infty)$ -MIN BP HRC

We now establish that the problem of deciding whether an instance of $(\infty, 1, \infty)$ -HRC admits a stable matching is NP-complete.

Theorem 8 *Given an instance of $(\infty, 1, \infty)$ -HRC, the problem of deciding whether there exists a stable matching is NP-complete. The result holds even if each hospital has capacity 1.*

Proof. The proof of this result uses a reduction from a restricted version of the vertex cover problem. More specifically, let VC3 denote the problem of deciding, given a cubic graph G and an integer K , whether G contains a vertex cover of size at most K . This problem is NP-complete [15, 22].

The problem of deciding whether there exists a stable matching in an instance of $(\infty, 1, \infty)$ -HRC is clearly in NP, as a given assignment may be verified to be a stable matching in polynomial time. To show NP-hardness, let $\langle G, K \rangle$ be an instance of VC3, where $G = (V, E)$, $E = \{e_1, \dots, e_m\}$ and $V = \{v_1, \dots, v_n\}$. For each i ($1 \leq i \leq n$), suppose that v_i is incident to edges e_{j_1} , e_{j_2} and e_{j_3} in G , where without loss of generality $j_1 < j_2 < j_3$. Define $e_{i,s} = e_{j_s}$ ($1 \leq s \leq 3$). Similarly, for each j ($1 \leq j \leq m$), suppose that $e_j = \{v_{i_1}, v_{i_2}\}$, where without loss of generality $i_1 < i_2$. Define $v_{j,r} = v_{i_r}$ ($1 \leq r \leq 2$).

We form an instance I of $(\infty, 1, \infty)$ -HRC as follows. The set of residents in I is $A \cup B \cup F \cup R \cup X \cup Y$ where $A = \{a_t : 1 \leq t \leq K\}$, $B = \{b_t : 1 \leq t \leq n - K\}$, $F = \bigcup_{t=1}^K F_t$, where $F_t = \{f_t^s : 1 \leq s \leq 6\}$, $R = \bigcup_{j=1}^m R_j$, where $R_j = \{r_j^s : 1 \leq s \leq 4\}$, $X = \{x_i : 1 \leq i \leq n\}$ and $Y = \bigcup_{t=1}^{n-K} Y_t$, where $Y_t = \{y_t^s : 1 \leq s \leq 6\}$.

The set of hospitals in I is $G \cup H \cup P \cup Q \cup Z$, where $G = \bigcup_{t=1}^K G_t$, where $G_t = \{g_t^r : 1 \leq r \leq 3\}$ ($1 \leq t \leq K$), $H = \bigcup_{j=1}^m H_j$, $H_j = \{h_j^s : 1 \leq s \leq 2\}$, $P = \{p_t : 1 \leq t \leq K\}$, $Q = \{q_t : 1 \leq t \leq n - K\}$ and $Z = \bigcup_{t=1}^{n-K} Z_t$, where $Z_t = \{z_t^r : 1 \leq r \leq 3\}$ and each hospital has capacity 1. The preference lists of the resident couples, single residents and hospitals in I are shown in Figure 7.

In the preference list of a resident x_i ($1 \leq i \leq n$) the symbol $h^s(x_i)$ ($1 \leq s \leq 3$) denotes the hospital h_j^r ($1 \leq r \leq 2$) such that $e_j = e_{i,s}$ and $v_i = v_{j,r}$. Similarly, in the preference list of a hospital h_j^r ($1 \leq j \leq m, 1 \leq r \leq 2$) the symbol $x(h_j^r)$ denotes the resident x_i such that $v_i = v_{j,r}$.

We claim that G contains a vertex cover of size at most K if and only if I admits a stable matching. Let C be a vertex cover in G such that $|C| \leq K$. Without loss of generality we may assume that $|C| = K$ for if otherwise a sufficient number of vertices can be added to C without violating the vertex cover condition.

We show how to define a matching M in I as follows. Let $C = \{v_{r_1}, v_{r_2}, \dots, v_{r_K}\}$ where without loss of generality $r_1 < r_2 < \dots < r_K$. Further let $V \setminus C = \{v_{s_1}, v_{s_2}, \dots, v_{s_{n-K}}\}$ where without loss of generality $s_1 < s_2 < \dots < s_{n-K}$. For each vertex $v_{r_i} \in C$ add the pairs $\{(x_{r_i}, p_i), (a_i, g_i), (f_i^3, g_i^2), (f_i^4, g_i^3)\}$ for $1 \leq i \leq K$ to M . For each vertex $v_{s_i} \in V \setminus C$ add $\{(x_{s_i}, q_i), (b_i, z_i), (y_i^3, z_i^2), (y_i^4, z_i^3)\}$ for $1 \leq i \leq n - K$ to M .

For each edge $e_j \in E$ at least one of $v_{j,1}$ or $v_{j,2}$ must be in C . If $v_{j,1} \in C$ add the pairs $\{(r_j^3, h_j^1), (r_j^4, h_j^2)\}$ to M . Otherwise $v_{j,2} \in C$ so add the pairs $\{(r_j^1, h_j^1), (r_j^2, h_j^2)\}$ to M .

We now show that M is a stable matching in I . Firstly, we show that no hospital $h_j^r \in H$ ($1 \leq j \leq m, 1 \leq r \leq 2$) can form part of a blocking pair

Residents' Preferences	
$(r_j^1, r_j^2) : (h_j^1, h_j^2)$	$(1 \leq j \leq m)$
$(r_j^3, r_j^4) : (h_j^1, h_j^2)$	$(1 \leq j \leq m)$
$(f_t^1, f_t^2) : (g_t^1, g_t^2)$	$(1 \leq t \leq K)$
$(f_t^3, f_t^4) : (g_t^2, g_t^3)$	$(1 \leq t \leq K)$
$(f_t^5, f_t^6) : (g_t^3, g_t^1)$	$(1 \leq t \leq K)$
$(y_t^1, y_t^2) : (z_t^1, z_t^2)$	$(1 \leq t \leq n - K)$
$(y_t^3, y_t^4) : (z_t^2, z_t^3)$	$(1 \leq t \leq n - K)$
$(y_t^5, y_t^6) : (z_t^3, z_t^1)$	$(1 \leq t \leq n - K)$
$a_t : p_t \quad g_t$	$(1 \leq t \leq K)$
$b_t : q_t \quad z_t$	$(1 \leq t \leq n - K)$
$x_i : p_1 \quad p_2 \quad \dots \quad p_K \quad h^1(x_i) \quad h^2(x_i) \quad h^3(x_i) \quad q_1 \quad q_2 \quad \dots \quad q_{n-K} \quad (1 \leq i \leq n)$	
Hospitals' Preferences	
$g_t^1 : a_t \quad f_t^1 \quad f_t^6$	$(1 \leq t \leq K)$
$g_t^2 : f_t^3 \quad f_t^2$	$(1 \leq t \leq K)$
$g_t^3 : f_t^5 \quad f_t^4$	$(1 \leq t \leq K)$
$h_j^1 : r_j^1 \quad x(h_j^1) \quad r_j^3$	$(1 \leq j \leq m)$
$h_j^2 : r_j^4 \quad x(h_j^2) \quad r_j^2$	$(1 \leq j \leq m)$
$p_t : x_1 \quad x_2 \quad \dots \quad x_n \quad a_t$	$(1 \leq t \leq K)$
$q_t : x_1 \quad x_2 \quad \dots \quad x_n \quad b_t$	$(1 \leq t \leq n - K)$
$z_t^1 : b_t \quad y_t^1 \quad y_t^6$	$(1 \leq t \leq n - K)$
$z_t^2 : y_t^3 \quad y_t^2$	$(1 \leq t \leq n - K)$
$z_t^3 : y_t^5 \quad y_t^4$	$(1 \leq t \leq n - K)$

Fig. 7. Preference lists in I , the constructed instance of $(\infty, 1, \infty)$ -HRC.

of M . Assume a hospital $h_j^r \in H$ is part of a blocking pair of M for some j ($1 \leq j \leq m$) and r ($1 \leq r \leq 2$). Now, since C is a vertex cover in G , an arbitrary edge $e_j \in E$ must be covered by either $v_{j,1}$ or $v_{j,2}$ or both. Assume firstly that $v_{j,1} \in C$. Then by construction $(x_{r_t}, p_t) \in M$ and $\{(r_j^3, h_j^1), (r_j^4, h_j^2)\} \subseteq M$ where $v_{j,1} = v_{r_t}$. Assume $(x(h_j^1), h_j^1)$ blocks M for some j ($1 \leq j \leq m$). Since $v_{j,1} \in C$ and thus $M(x(h_j^1)) \in P$, $x(h_j^1)$ prefers $M(x(h_j^1))$ to h_j^1 , a contradiction. Now assume that $((r_j^1, r_j^2), (h_j^1, h_j^2))$ blocks M . However, h_j^2 prefers $M(h_j^2) = r_j^4$ to r_j^2 , a contradiction.

Now assume $v_{j,1} \notin C$. Then $v_{j,2} \in C$ and by construction $(x_{r_{t'}}, p_{t'}) \in M$ and $\{(r_j^1, h_j^1), (r_j^2, h_j^2)\} \subseteq M$ where $v_{j,2} = v_{r_{t'}}$. Assume $(x(h_j^2), h_j^2)$ blocks M for some j ($1 \leq j \leq m$). Since $v_{j,2} \in C$ and thus $M(x(h_j^2)) \in P$, $x(h_j^2)$ prefers $M(x(h_j^2))$ to h_j^2 , a contradiction. Now assume $((r_j^3, r_j^4), (h_j^1, h_j^2))$ blocks M . However, h_j^1 prefers $M(h_j^1) = r_j^1$ to r_j^3 , a contradiction. Thus, no $h_j^r \in H$ ($1 \leq j \leq m, 1 \leq r \leq 2$) can form part of a blocking pair of M .

We now show that no hospital in $P \cup Q$ can be involved in a blocking pair of M . By construction $M(p_t) \in X$ for all t ($1 \leq t \leq K$). Assume some pair (x_{k_1}, p_{l_1}) blocks M . Let $M(x_{k_1}) = p_{l_2}$ and $M(p_{l_1}) = x_{k_2}$. Since (x_{k_1}, p_{l_1}) blocks M then $l_1 < l_2$ and $k_1 < k_2$ in contradiction to the construction of M . A similar argument shows that no hospital in Q may be involved in a blocking pair of M and thus we have that no hospital in $P \cup Q$ may be involved in a blocking pair of M .

We now show that no hospital in $G \cup Z$ can be involved in a blocking pair of M . Firstly, assume a hospital $g_t^s \in H$ is part of a blocking pair of M for some t ($1 \leq t \leq K$) and s ($1 \leq s \leq 3$). Clearly, since g_t^1 and g_t^2 are both assigned their first preference they cannot form part of a blocking pair for M . Hospital g_t^3 prefers f_t^5 to $M(g_t^3) = f_t^4$. However, f_t^5 is a member of the couple (f_t^5, f_t^6) that expresses a joint preference for the pair (g_t^3, g_t^1) and g_t^1 prefers $M(g_t^1) = a_t$ to f_t^6 , a contradiction. Thus, no hospital $g_t^s \in H$ ($1 \leq t \leq K, 1 \leq s \leq 3$) can form part of a blocking pair of M . A similar argument may be used to show that no $z_t^s \in H$ ($1 \leq t \leq n - K, 1 \leq s \leq 3$) can form part of a blocking pair of M and thus we have that no hospital in $G \cup Z$ can be involved in a blocking pair of M .

We now have that no hospital in I may be part of a blocking pair of M and thus M must be stable.

Conversely, let M be a stable matching in I . We first show that the stability of M implies that $M(x_i) \in P \cup Q$ for all i ($1 \leq i \leq n$). Observe that if $(a_t, g_t^1) \notin M$ for t ($1 \leq t \leq K$) then no stable assignment is possible amongst the agents in $F_t \cup G_t$ as shown in Lemma 13. However, if $\{(a_t, g_t^1), (f_t^3, g_t^2), (f_t^4, g_t^3)\} \subseteq M$ then no blocking pair exists in $F_t \cup G_t$. It follows that if $(a_t, g_t^1) \in M$ then (a_t, p_t) blocks M unless $M(p_t) \in X$. A similar argument shows that $M(q_t) \in X$ for all t ($1 \leq t \leq n - K$). Now, since $|X| = n$ and $|P \cup Q| = n$, clearly all $x \in X$ must be partnered with a member of $P \cup Q$ and moreover, $M(x_i) \notin H$ in any stable matching in I .

Next we show that the stability of M implies that h_j^1 and h_j^2 are fully subscribed in M for all j ($1 \leq j \leq m$). Let j ($1 \leq j \leq m$) be given. Assume

that both h_j^1 and h_j^2 are undersubscribed in M . Since $M(x(h_j^r)) \neq h_j^r$ for all j, r ($1 \leq j \leq m, 1 \leq r \leq 2$), $((r_j^1, r_j^2), (h_j^1, h_j^2))$ blocks M , a contradiction. Thus either $\{(r_j^1, h_j^1), (r_j^2, h_j^2)\} \subseteq M$ or $\{(r_j^3, h_j^1), (r_j^4, h_j^2)\} \subseteq M$ in any stable matching in I . If $\{(r_j^1, h_j^1), (r_j^2, h_j^2)\} \subseteq M$ then $((r_j^3, r_j^4), (h_j^1, h_j^2))$ does not block M . Similarly, if $\{(r_j^3, h_j^1), (r_j^4, h_j^2)\} \subseteq M$ then $((r_j^1, r_j^2), (h_j^1, h_j^2))$ does not block M . Thus we have that h_j^1 and h_j^2 are fully subscribed in M for all j ($1 \leq j \leq m$). Moreover, we have that all hospitals must be fully subscribed in any stable matching M in I .

Define a set of vertices C in G as follows. For each i ($1 \leq i \leq n$) if $M(x_i) \in P$, add v_i to C . Since M is a stable matching and $|P| = K$, this process selects exactly K of the n vertices in V and thus $|C| = K$. We now show that C represents a vertex cover in G . Consider an arbitrary edge $e_j \in E$. Assume that both $v_{j,1} \notin C$ and $v_{j,2} \notin C$ and hence that C is not a vertex cover in G . Then $M(x_{j,1}) \in Q$ and $M(x_{j,2}) \in Q$. As M is stable and thus hospital complete, either $\{(r_j^1, h_j^1), (r_j^2, h_j^2)\} \subset M$ or $\{(r_j^3, h_j^1), (r_j^4, h_j^2)\} \subset M$. If $\{(r_j^1, h_j^1), (r_j^2, h_j^2)\} \subset M$ then $(x_{j,2}, h_j^2)$ blocks M , a contradiction. If $\{(r_j^3, h_j^1), (r_j^4, h_j^2)\} \subset M$ then $(x_{j,1}, h_j^1)$ blocks M , a contradiction. Hence C represents a vertex cover in G of size K and the theorem is proven. \square

Corollary 9 $(\infty, 1, \infty)$ -MIN BP HRC is NP-hard and not approximable within a factor of $n_1^{1-\varepsilon}$, for any $\varepsilon > 0$, unless $P=NP$, where n_1 is the number of residents in a given instance. The result holds even if each hospital has capacity 1.

Proof. The proof of this result is analogous to the proof of Theorem 2 in [7], which establishes the same result for $(0, 2, 2)$ -MIN BP HRC, using the NP-completeness of the problem of deciding whether a stable matching exists in a given instance of $(0, 2, 2)$ -HRC as a starting point. The restrictions on the preference list lengths are not used in the gap-introducing reduction that proves the inapproximability result, hence the same reduction can be used to demonstrate the inapproximability of $(\infty, 1, \infty)$ -MIN BP HRC. \square

C Efficiently solvable variants of MIN BP HRC

C.1 Fixed assignments in HRC

In an instance I of HRC some agents may rank one another highly in their preference lists, leading to the outcome that they must be assigned to one another in any stable matching in I . We describe these agents as *fixed assignments*, using the following lemma to define this concept formally and to show that fixed assignments must belong to any stable matching in I .

Lemma 10 *Let I be an arbitrary instance of HRC.*

- (i) *If a single resident r_i has a hospital h_j in first place on his preference list and r_i is within the first c_j places on h_j 's preference list then (r_i, h_j) must belong to any stable matching in I .*

- (ii) If a couple (r_i, r_j) has a hospital pair (h_p, h_q) in first place on its joint preference list and r_i is within the first c_p places on h_p 's preference list and also r_j is within the first c_q places on h_q 's preference list then (r_i, r_j) must be jointly assigned to (h_p, h_q) in any stable matching in I .

Any pair consisting of a single resident and a single hospital satisfying Case (i), or consisting of a couple and hospital pair satisfying Case (ii), is called a fixed assignment in I .

Proof. The proof of the Lemma follows immediately from the fact that any matching M in which r_i is not assigned to h_j will be blocked by (r_i, h_j) , and similarly, any matching M in which (r_i, r_j) and (h_p, h_q) are not assigned to each other will be blocked by (r_i, r_j) with (h_p, h_q) . \square

Suppose that a matching M is constructed solely by matching agents who are involved in fixed assignments. As a consequence, suppose some pair (r_i, h_j) is added to M . Clearly no other hospital may be assigned in M to r_i , and hence r_i can be deleted from the preference list of each other hospital in which he appears. Moreover, in the event that h_j becomes fully subscribed by accepting r_i as an assignee, h_j can be deleted from the preference list of each resident other than r_i in which it appears. We say that we *satisfy* a fixed assignment if we match together in M the agents involved, and then carry out the corresponding deletions as described above. Note that making these deletions may expose another fixed assignment in the resulting reduced instance of HRC, which can then also be satisfied in M . If we continue satisfying fixed assignments until no more fixed assignments are exposed then we say all fixed assignments have been *iteratively satisfied* in M . This idea is used in the proof of the following proposition, to show that a stable solution can be found in an instance of $(\infty, \infty, 1)$ -HRC in polynomial time.

Proposition 11 *An instance I of $(\infty, \infty, 1)$ -HRC admits exactly one stable matching, which can be found in polynomial time.*

Proof. Consider an arbitrary single resident r_i in I . Let the hospital in first place on resident r_i 's preference list be h_j . Since r_i must be in first place in h_j 's preference list (as it is the only preference expressed by h_j), the pair (r_i, h_j) represents a fixed assignment in I . Thus, any single resident in I must be part of exactly one fixed assignment in I and this may be satisfied by assigning each single resident to the hospital in first place on his preference list.

Now, consider an arbitrary couple (r_i, r_j) in I . Let the hospital pair (h_p, h_q) be in first place on couple (r_i, r_j) 's joint preference list. Clearly, since r_i (respectively r_j) is in first place on h_p 's (respectively h_q 's) preference list, (r_i, r_j) with (h_p, h_q) represents a fixed assignment in I . Thus, any resident couple in I must be part of exactly one fixed assignment in I and this may be satisfied by assigning each couple to the hospital pair in first place on their joint preference list.

Hence, the fixed assignments involving both the single residents and the couples in I may be satisfied iteratively in time linear in the number of residents

in I , leading to a matching M that is clearly stable in I , and which is the only stable matching in I . \square

C.2 (2, 1, 2)-MIN BP HRC is efficiently solvable

Let I be an instance of (2, 1, 2)-HRC, and assume that M_0 is a matching in I in which all fixed assignments have been iteratively satisfied, and assume that the corresponding deletions have been made from the preference lists in I . In Lemma 12 below, we use the absence of fixed assignments in I to infer that I must be constructed from the union of a finite number of disjoint discrete sub-instances of (2, 1, 2)-HRC and further that each disjoint sub-instance I' of I must be of the form shown in Figure 8. Let I' be one of these disjoint sub-instances of I . We prove in Lemma 13 that the number of couples involved in I' determines whether I' admits a stable matching; indeed, I' admits a stable matching if and only if the number of couples in I' is even.

Lemma 12 *An arbitrary instance of (2, 1, 2)-HRC involving at least one couple and in which all fixed assignments have been iteratively satisfied must be constructed from sub-instances of the form shown in Figure 8 in which all of the hospitals have capacity 1.*

Proof. Let I be an arbitrary instance of (2, 1, 2)-HRC in which all fixed assignments have been iteratively satisfied. Observe that if a couple expresses a preference for a hospital pair (h_p, h_p) this would represent a fixed assignment, a contradiction. Thus, no couple may express such a preference in I . We now show how the absence of fixed assignments in I allows us to infer the preference lists for all of the agents involved in I .

Let $(r_{c_1}^1, r_{c_1}^2)$ be a couple in I and further let $(h_{c_1}^0, h_{c_1}^1)$ be the hospital pair for which $(r_{c_1}^1, r_{c_1}^2)$ expresses a preference. Since all fixed assignments have been iteratively satisfied by construction, it cannot be the case that *both*:

- (i) $h_{c_1}^0$ has capacity two or has $r_{c_1}^1$ in first place in its preference list *and*
- (ii) $h_{c_1}^1$ has capacity two or has $r_{c_1}^2$ in first place in its preference list.

Without loss of generality, assume that $h_{c_1}^1$ has capacity one and does not have $r_{c_1}^2$ in first place in its preference list. Hence there exists some other resident r_x who is preferred by $h_{c_1}^1$. Clearly, this resident is either a member of a couple or is a single resident. We now consider both of these cases and show that we must arrive at the same outcome in either case. In what follows n_k ($1 \leq k \leq n_N$) represents the number of single residents generated following couple c_k as the preference lists of the residents are inferred in the proof.

Case (i): r_x is single and thus $n_1 > 0$. In this case let $r_x = r_{s_1}^1$. Since $r_{s_1}^1$ is in first place in the preference list of $h_{c_1}^1$, to prevent a fixed assignment, there must exist another hospital that is preferred by $r_{s_1}^1$; let this be $h_{c_1}^2$. If $h_{c_1}^2$ has capacity two then $(r_{s_1}^1, h_{c_1}^2)$ represents a fixed assignment, a contradiction. Hence, $h_{c_1}^2$ must have capacity one.

Residents	Hospitals
$(r_{c_1}^1, r_{c_1}^2) : (h_{c_1}^0, h_{c_1}^1)$	$h_{c_1}^0 : r_{c_1}^1 \quad r_{s_N}^{n_N}$
$r_{s_1}^1 : h_{c_1}^2 \quad h_{c_1}^1$	$h_{c_1}^1 : r_{s_1}^1 \quad r_{c_1}^2$
$r_{s_1}^2 : h_{c_1}^3 \quad h_{c_1}^2$	$h_{c_1}^2 : r_{s_1}^2 \quad r_{s_1}^1$
\vdots	\vdots
$r_{s_1}^{n_1} : h_{c_1}^{n_1+1} \quad h_{c_1}^{n_1}$	$h_{c_1}^{n_1} : r_{s_1}^{n_1} \quad r_{s_1}^{n_1-1}$
$(r_{c_2}^1, r_{c_2}^2) : (h_{c_1}^{n_1+1}, h_{c_2}^1)$	$h_{c_1}^{n_1+1} : r_{c_2}^1 \quad r_{s_1}^{n_1}$
$r_{s_2}^1 : h_{c_2}^2 \quad h_{c_2}^1$	$h_{c_2}^1 : r_{s_2}^1 \quad r_{c_2}^2$
$r_{s_2}^2 : h_{c_2}^3 \quad h_{c_2}^2$	$h_{c_2}^2 : r_{s_2}^2 \quad r_{s_2}^1$
\vdots	\vdots
$r_{s_2}^{n_2} : h_{c_2}^{n_2+1} \quad h_{c_2}^{n_2}$	$h_{c_2}^{n_2} : r_{s_2}^{n_2} \quad r_{c_2}^{n_2-1}$
$(r_{c_3}^1, r_{c_3}^2) : (h_{c_2}^{n_2+1}, h_{c_3}^1)$	$h_{c_2}^{n_2+1} : r_{c_3}^1 \quad r_{s_2}^{n_2}$
$r_{s_3}^1 : h_{c_3}^2 \quad h_{c_3}^1$	$h_{c_3}^1 : r_{s_3}^1 \quad r_{c_3}^2$
$r_{s_3}^2 : h_{c_3}^3 \quad h_{c_3}^2$	$h_{c_3}^2 : r_{s_3}^2 \quad r_{s_3}^1$
\vdots	\vdots
$r_{s_{N-1}}^{n_{N-1}} : h_{c_{N-1}}^{n_{N-1}+1} \quad h_{c_{N-1}}^{n_{N-1}}$	$h_{c_{N-1}}^{n_{N-1}} : r_{s_{N-1}}^{n_{N-1}} \quad r_{c_{N-1}}^{n_{N-1}-1}$
$(r_{c_N}^1, r_{c_N}^2) : (h_{c_{N-1}}^{n_{N-1}+1}, h_{c_N}^1)$	$h_{c_{N-1}}^{n_{N-1}+1} : r_{c_N}^1 \quad r_{s_{N-1}}^{n_{N-1}}$
$r_{s_N}^1 : h_{c_N}^2 \quad h_{c_N}^1$	$h_{c_N}^1 : r_{s_N}^1 \quad r_{c_N}^2$
$r_{s_N}^2 : h_{c_N}^3 \quad h_{c_N}^2$	$h_{c_N}^2 : r_{s_N}^2 \quad r_{s_N}^1$
\vdots	\vdots
$r_{s_N}^{n_N} : h_{c_1}^0 \quad h_{c_N}^{n_N}$	$h_{c_N}^{n_N} : r_{s_N}^{n_N} \quad r_{s_N}^{n_N-1}$

Fig. 8. An instance of $(2, 1, 2)$ -HRC containing an arbitrary number of couples and an arbitrary number of residents that has no unsatisfied fixed assignments.

Now, since $r_{s_1}^1$ has $h_{c_1}^2$ in first place in its preference list, there must exist some other resident who is preferred by $h_{c_1}^2$. We consider first the case where each newly generated resident is single. Hence, let this new resident be a single resident, $r_{s_1}^2$. Since $r_{s_1}^2$ is in first place on the preference list of $h_{c_1}^2$ there must exist another hospital which is preferred by $r_{s_1}^2$; let this new hospital be $h_{c_1}^3$. Assume $h_{c_1}^3$ has capacity two. In that case $(r_{s_1}^1, h_{c_1}^2)$ represents a fixed assignment, a contradiction. Hence, $h_{c_1}^3$ must have capacity one.

We may continue constructing a sequence of distinct single residents and hospitals of capacity one, but as the number of single residents is finite, ultimately we must eventually arrive at a resident who is a member of a couple; let this resident be $r_{c_2}^1$. Without loss of generality suppose that $r_{c_2}^1$ is the first member of the couple to which he belongs. Let $r_{s_1}^{n_1}$ be the final single resident constructed in the preceding sequence.

It follows that $r_{s_1}^{n_1}$ prefers some hospital $h_{c_1}^{n_1+1}$ of capacity one to $h_{c_1}^{n_1}$. If $h_{c_1}^{n_1+1} = h_{c_1}^0$ then I contains precisely one couple and the instance is of the form

Residents			
$(r_{c_1}^1, r_{c_1}^2)$:	$(h_{c_1}^0, h_{c_1}^1)$	
$(r_{c_k}^1, r_{c_k}^2)$:	$(h_{c_{k-1}}^{n_{k-1}+1}, h_{c_k}^1)$	$2 \leq k \leq N-1$
$(r_{c_N}^1, r_{c_N}^2)$:	$(h_{c_{N-1}}^{n_{N-1}+1}, h_{c_N}^1)$	$n_N > 0$
$(r_{c_N}^1, r_{c_N}^2)$:	$(h_{c_{N-1}}^{n_{N-1}+1}, h_{c_1}^0)$	$n_N = 0$
$r_{s_k}^a$:	$h_{c_k}^{a+1} \quad h_{c_k}^a$	$1 \leq k \leq N, 1 \leq a \leq n_k, n_k > 0$
Hospitals			
$h_{c_1}^0$:	$r_{c_1}^1 \quad r_{s_N}^{n_N}$	if $n_N > 0$
$h_{c_1}^0$:	$r_{c_1}^1 \quad r_{c_N}^2$	if $n_N = 0$
$h_{c_k}^1$:	$r_{s_k}^1 \quad r_{c_k}^2$	$1 \leq k \leq N$, if $n_k > 0$
$h_{c_k}^1$:	$r_{c_{k+1}}^1 \quad r_{c_k}^2$	$1 \leq k \leq N$, if $n_k = 0$
$h_{c_k}^a$:	$r_{s_k}^a \quad r_{s_k}^{a-1}$	$1 \leq k \leq N, 2 \leq a \leq n_k$, if $n_k > 0$
$h_{c_k}^{n_k+1}$:	$r_{c_{k+1}}^1 \quad r_{s_k}^{n_k}$	$1 \leq k \leq N-1$, if $n_k > 0$

Fig. 9. An exactly equivalent description of the instance shown in Figure 8

shown in Figure 8 where $N = 1$ and $n_1 > 0$. Otherwise $h_{c_1}^{n_1+1}$ is a new hospital of capacity one that prefers $r_{c_2}^1$ to $r_{s_1}^{n_1}$. Since $h_{c_1}^{n_1+1}$ has $r_{c_2}^1$ in first place on its preference list, it must be the case that $r_{c_2}^1$ expresses a joint preference as part of the couple $(r_{c_2}^1, r_{c_2}^2)$ for a hospital pair involving $h_{c_1}^{n_1+1}$; let this pair be $(h_{c_1}^{n_1+1}, h_{c_2}^1)$. Since $h_{c_1}^{n_1+1}$ has $r_{c_2}^1$ in first place on its preference list, $h_{c_2}^1$ must be of capacity one and prefer some other resident to $r_{c_2}^2$, otherwise $(r_{c_2}^1, r_{c_2}^2)$ represents a fixed assignment with $(h_{c_1}^{n_1+1}, h_{c_2}^1)$, a contradiction. Now, let this other resident be r_y .

Case (ii): r_x is a member of a couple and thus $n_1 = 0$. Let $r_x = r_{c_2}^1$. Then $h_{c_1}^1$ prefers $r_{c_2}^1$ to $r_{c_1}^2$. Assume that $r_{c_2}^1$ is part of a couple $(r_{c_2}^1, r_{c_2}^2)$ and further assume that $(r_{c_2}^1, r_{c_2}^2)$ finds $(h_{c_1}^1, h_{c_2}^1)$ acceptable. If $h_{c_2}^1 = h_{c_1}^0$ then I contains exactly two couples and is of the form shown in Figure 8 with $N = 2$ and $n_1 = n_2 = 0$. (In this case $h_{c_1}^0$ prefers $r_{c_1}^1$ to $r_{c_2}^2$.) Otherwise, $h_{c_2}^1$ is a new hospital which must be of capacity one, or $(r_{c_2}^1, r_{c_2}^2)$ represents a fixed assignment with $(h_{c_1}^1, h_{c_2}^1)$, and moreover $h_{c_2}^1$ must prefer some other resident to $r_{c_2}^2$; let this resident be r_y .

Thus in both cases we have that if $h_{c_2}^1 \neq h_{c_1}^0$ then $h_{c_2}^1$ is of capacity one and prefers some resident r_y to $r_{c_2}^2$. Clearly, r_y is either a member of a couple or is a single resident. As before, we consider both of these cases and show that we must arrive at the same outcome in either case.

Case (i): r_y is single and thus $n_2 > 0$; In this case let $r_y = r_{s_2}^1$. Since $r_{s_2}^1$ is in first place on the preference list of $h_{c_2}^1$, it follows that $h_{c_2}^1$ cannot be in first place in the preference list of $r_{s_2}^1$. Hence, there must exist another hospital preferred by $r_{s_2}^1$; let this be $h_{c_2}^2$. Further, $h_{c_2}^2$ must be of capacity one and have a resident other than $r_{s_2}^1$ in first place in its preference list; let this resident be $r_{s_2}^2$. We consider first the case where each newly generated resident is single. Suppose $r_{s_2}^2$ is single. Since $r_{s_2}^2$ is in first place on the preference list of $h_{c_2}^2$ there must exist another hospital which is preferred by $r_{s_2}^2$; let this new hospital be $h_{c_2}^3$. Hospital $h_{c_2}^3$ must have capacity one, otherwise $(r_{s_2}^2, h_{c_2}^3)$ would represent a fixed assignment.

We may continue generating a sequence of distinct single residents and hospitals of capacity one, but since the number of residents is finite, we must eventually arrive at a resident who is a member of a couple; let this resident be $r_{c_3}^1$. Without loss of generality suppose that $r_{c_3}^1$ is the first member of the couple to which he belongs. Let $r_{s_2}^{n_2}$ be the final single resident in the previously generated sequence. Then $r_{s_2}^{n_2}$ prefers some hospital $h_{c_2}^{n_2+1}$ to $h_{c_2}^{n_2}$ and $h_{c_2}^{n_2+1}$ must be of capacity one. If $h_{c_2}^{n_2+1} = h_{c_1}^0$ then I contains precisely two couples. Otherwise $h_{c_2}^{n_2+1}$ is a new hospital of capacity one and prefers $r_{c_3}^1$ to $r_{s_2}^{n_2}$.

Since $h_{c_2}^{n_2+1}$ has $r_{c_3}^1$ in first place on its preference list, it must be the case that $r_{c_3}^1$ expresses a joint preference as part of the couple $(r_{c_3}^1, r_{c_3}^2)$ for a hospital pair involving $h_{c_2}^{n_2+1}$; let this pair be $(h_{c_2}^{n_2+1}, h_{c_3}^1)$.

Since $h_{c_3}^1$ has $r_{c_3}^2$ in first place on its preference list, $h_{c_3}^2$ must be of capacity one and prefer some other resident to $r_{c_3}^2$; let this resident be r_z .

Case (ii): r_y is a member of a couple and thus $n_2 = 0$. Let $r_y = r_{c_2}^1$. Then $h_{c_2}^1$ prefers $r_{c_3}^1$ to $r_{c_2}^2$. Assume that $r_{c_3}^1$ is part of a couple $(r_{c_3}^1, r_{c_3}^2)$ and further assume that $(r_{c_3}^1, r_{c_3}^2)$ finds $(h_{c_2}^1, h_{c_3}^1)$ acceptable. If $h_{c_3}^1 = h_{c_1}^0$ then I contains three couples and is of the form shown in Figure 8 with $N = 3$ and $n_3 = 0$. (In this case $h_{c_1}^0$ prefers $r_{c_1}^1$ to $r_{c_3}^2$.) Otherwise, $h_{c_3}^1$ is a new hospital which must be of capacity one (or else $(r_{c_3}^1, r_{c_3}^2)$ represents a fixed assignment with $(h_{c_2}^1, h_{c_3}^1)$) and must prefer some resident to $r_{c_3}^2$; let this resident be r_z .

Now, in both cases we have that if $h_{c_3}^1 \neq h_{c_1}^0$ then $h_{c_3}^1$ is of capacity one and prefers some resident r_z to $r_{c_3}^2$. As before, we may continue generating a sequence of distinct residents, couples and hospitals in this fashion, but since the number of residents and couples is finite, we must eventually reach some resident, either single or a member of a couple who must be in second place in $h_{c_1}^0$'s preference list and a complete instance of (2, 1, 2)-HRC is formed. Thus, the instance I must be of the form shown in Figure 8. \square

Lemma 13 *An instance I of (2, 1, 2)-HRC of the form shown in Figure 8 admits a stable matching if and only if the number of couples involved in I is even.*

Proof. Let M be a stable matching in I . It is either the case that $(r_{c_1}^1, r_{c_1}^2)$ is assigned in M or $(r_{c_1}^1, r_{c_1}^2)$ is unassigned in M . We now consider each of these cases and show that in either case if I contains an odd number of couples then I cannot admit a stable matching.

Case (i): Assume $(r_{c_1}^1, r_{c_1}^2)$ is assigned in M and therefore $(r_{c_1}^1, h_{c_1}^0) \in M$. Clearly either $n_1 = 0$ or $n_1 > 0$. We now show that whether $n_1 = 0$ or $n_1 > 0$, if $(r_{c_1}^1, r_{c_1}^2)$ is assigned in M then $(r_{c_2}^1, r_{c_2}^2)$ is unassigned in M .

If $n_1 = 0$ and the instance contains exactly one couple, then $(r_{c_1}^1, r_{c_1}^2)$ represents a fixed assignment with $(h_{c_1}^0, h_{c_1}^1)$, a contradiction. Thus, I contains more than one couple. Let the second couple in I be $(r_{c_2}^1, r_{c_2}^2)$ such that $h_{c_1}^1$ has $r_{c_2}^1$ in first place on its preference list. We now have that $(r_{c_2}^1, r_{c_2}^2)$ expresses a preference for $(h_{c_1}^1, h_{c_2}^1)$ and since $(r_{c_1}^2, h_{c_1}^1) \in M$, clearly $(r_{c_2}^1, r_{c_2}^2)$ cannot be assigned in M .

If $n_1 > 0$ then $h_{c_1}^1$ has $r_{s_1}^1$ in first place on its preference list. Now, if $r_{s_1}^1$ is unassigned in M then $(r_{s_1}^1, h_{c_1}^1)$ blocks M . Hence $r_{s_1}^1$ must be assigned in M and moreover $(r_{s_1}^1, h_{c_1}^2) \in M$. In similar fashion we may confirm that each $r_{s_1}^a$ ($1 \leq a < n_1$) is assigned to the hospital $h_{c_1}^{a+1}$ in first place on its preference list.

Now consider, $r_{s_1}^{n_1}$. Again $r_{s_1}^{n_1}$ must be assigned to the hospital in first place in its preference list. If I contains exactly one couple then this hospital must be $h_{c_1}^0$ by Lemma 12. However, by assumption $(r_{c_1}^1, h_{c_1}^0) \in M$, a contradiction. Thus I must contain more than one couple. Now, let $h_{c_1}^{n_1+1}$ be the hospital in first place on $r_{s_1}^{n_1}$'s preference list. Since $(r_{s_1}^{n_1}, h_{c_1}^{n_1+1}) \in M$, clearly $(r_{c_2}^1, r_{c_2}^2)$ cannot be assigned in M as the only pair they find acceptable is $(h_{c_1}^{n_1+1}, h_{c_2}^1)$. Thus, we have that whether $n_1 = 0$ or $n_1 > 0$, if $(r_{c_1}^1, r_{c_1}^2)$ is assigned in M then $(r_{c_2}^1, r_{c_2}^2)$ is not assigned in M .

Now, either $n_2 = 0$ or $n_2 > 0$. We now show that whether $n_2 = 0$ or $n_2 > 0$, if $(r_{c_2}^1, r_{c_2}^2)$ is unassigned in M then $(r_{c_3}^1, r_{c_3}^2)$ must be assigned in M . If $n_2 = 0$ and the instance contains exactly two couples then $(r_{c_2}^1, r_{c_2}^2)$ expresses a preference for either $(h_{c_1}^1, h_{c_1}^0)$ if $n_1 = 0$ (or $(h_{c_1}^{n_1+1}, h_{c_1}^0)$ if $n_1 > 0$) and $h_{c_1}^0$ has $r_{c_2}^2$ in second place on its preference list. In this case, the instance admits exactly two stable matchings of equal cardinality. If $n_2 = 0$ and the instance contains more than two couples then $(r_{c_2}^1, r_{c_2}^2)$ expresses a preference for $(h_{c_2}^1, h_{c_3}^1)$. Now assume, $h_{c_2}^1$ is unassigned in M . Then $(r_{c_2}^1, r_{c_2}^2)$ blocks M with $(h_{c_1}^1, h_{c_1}^0)$ if $n_1 = 0$ (or $(h_{c_1}^{n_1+1}, h_{c_1}^0)$ if $n_1 > 0$), a contradiction. Thus we have that if $(r_{c_2}^1, r_{c_2}^2)$ is not assigned in M then $(r_{c_3}^1, r_{c_3}^2)$ must be assigned to $(h_{c_2}^1, h_{c_3}^1)$ in M .

If $n_2 > 0$ then $h_{c_2}^1$ has $r_{s_2}^1$ in first place on its preference list. Now, if $r_{s_2}^1$ is not assigned in M then $(r_{s_2}^1, h_{c_2}^1)$ blocks M , a contradiction. Hence $r_{s_2}^1$ must be assigned in M and moreover $(r_{s_2}^1, h_{c_2}^2) \in M$. In similar fashion we may confirm that each $r_{s_2}^a$ ($1 \leq a \leq n_2$) must be assigned in M to the hospital $h_{s_2}^{a+1}$ in first place in its preference list.

Now consider, $r_{s_2}^{n_2}$. If the instance contains exactly two couples then the hospital in first place in the preference list of $r_{s_2}^{n_2}$ must be $h_{c_1}^0$ and the result follows. However, if the instance contains more than two couples then the hospital in first place in the preference list of $r_{s_2}^{n_2}$ must be a new hospital $h_{c_2}^{n_2+1}$. Now let the next couple be $(r_{c_3}^1, r_{c_3}^2)$. Assume $(r_{c_3}^1, r_{c_3}^2)$ is unassigned in M . Then $(r_{s_2}^{n_2}, h_{c_2}^{n_2+1})$ must block M , so $(r_{c_3}^1, r_{c_3}^2)$ must be assigned to $(h_{c_2}^{n_2+1}, h_{c_3}^1)$ in M . Thus, whether $n_2 = 0$ or $n_2 > 0$, if $(r_{c_2}^1, r_{c_2}^2)$ is unassigned in M then $(r_{c_3}^1, r_{c_3}^2)$ must be assigned in M .

In similar fashion either $n_3 = 0$ or $n_3 > 0$. Again, we show that whether $n_3 = 0$ or $n_3 > 0$, if $(r_{c_3}^1, r_{c_3}^2)$ is assigned in M then $(r_{c_4}^1, r_{c_4}^2)$ is not assigned in M . If $n_3 = 0$ and the instance contains exactly three couples then $(r_{c_3}^1, r_{c_3}^2)$ is assigned to $(h_{c_2}^1, h_{c_1}^0)$ if $n_2 = 0$ (or $(h_{c_1}^{n_2+1}, h_{c_1}^0)$ if $n_2 > 0$) and $h_{c_1}^0$ has $r_{c_3}^2$ in second place on its preference list. However, by assumption $(r_{c_1}^2, h_{c_1}^0) \in M$, a contradiction. Thus, I contains more than three couples and $(r_{c_4}^1, r_{c_4}^2)$ expresses a preference for $(h_{c_3}^1, h_{c_4}^1)$ and since $(r_{c_3}^1, h_{c_3}^1) \in M$, $(r_{c_4}^1, r_{c_4}^2)$ cannot be assigned in M .

If $n_3 > 0$ then $h_{c_3}^1$ has $r_{s_3}^1$ in first place on its preference list. Now, if $r_{s_3}^1$ is not assigned in M then $(r_{s_3}^1, h_{c_3}^1)$ blocks M , a contradiction. Hence $r_{s_3}^1$ must be assigned in M and moreover $(r_{s_3}^1, h_{c_3}^1) \in M$. In similar fashion we may confirm that each $r_{s_3}^a$ ($1 \leq a < n_3$) is assigned to the hospital $h_{c_3}^{a+1}$ in first place on its preference list.

Now consider $r_{s_3}^{n_3}$. If the instance contains exactly three couples then the hospital in first place in the preference list of $r_{s_3}^{n_3}$ must be $h_{c_1}^0$. However, by construction, $(r_{c_1}^2, h_{c_1}^0) \in M$, a contradiction. Hence, the instance must have more than three couples and the hospital in first place in the preference list of $r_{s_3}^{n_3}$ must be a new hospital $h_{c_3}^{n_3+1}$. Now let the next couple be $(r_{c_4}^1, r_{c_4}^2)$. Since $(r_{s_3}^{n_3}, h_{c_3}^{n_3+1}) \in M$, $(r_{c_4}^1, r_{c_4}^2)$ cannot be assigned in M . Thus, whether $n_3 = 0$ or $n_3 > 0$, if $(r_{c_3}^1, r_{c_3}^2)$ is assigned in M then $(r_{c_4}^1, r_{c_4}^2)$ is not assigned in M .

Finally we consider whether $n_4 = 0$ or $n_4 > 0$. If $n_4 = 0$ and the instance contains exactly four couples then $(r_{c_4}^1, r_{c_4}^2)$ expresses a preference for the hospital pair $(h_{c_4}^1, h_{c_1}^0)$ and $h_{c_1}^0$ has $r_{c_4}^2$ in second place on its preference list and the result follows. Otherwise the instance contains more than four couples and $(r_{c_5}^1, r_{c_5}^2)$ expresses a preference for $(h_{c_4}^1, h_{c_5}^1)$. Now assume, $h_{c_4}^1$ is unassigned in M . Then $(r_{c_4}^1, r_{c_4}^2)$ blocks M with $(h_{c_4}^{n_4+1}, h_{c_4}^1)$, a contradiction. Thus $(r_{c_5}^1, r_{c_5}^2)$ must be assigned to $(h_{c_4}^1, h_{c_5}^1)$ in M .

If $n_4 > 0$ then $h_{c_4}^1$ has $r_{s_4}^1$ in first place on its preference list. If $r_{s_4}^1$ is not assigned in M then $(r_{s_4}^1, h_{c_4}^1)$ blocks M , a contradiction. Hence $r_{s_4}^1$ must be assigned in M and moreover $(r_{s_4}^1, h_{c_4}^1) \in M$. In similar fashion we may confirm that each $r_{s_4}^a$ ($1 \leq a \leq n_4$) must be assigned in M to the hospital $h_{s_4}^{a+1}$ in first place in its preference list. Now consider, $r_{s_4}^{n_4}$. If the instance contains exactly four couples then the hospital in first place in the preference list of $r_{s_4}^{n_4}$ must be $h_{c_1}^0$ and the result follows.

At this point we observe that argument is similar for the case that the number of couples is larger than four. As the preceding argument shows, if the number of couples is odd, then no stable matching exists, a contradiction.

Case (ii): Now suppose that $(r_{c_1}^1, r_{c_1}^2)$ is unassigned in M . Then essentially $(r_{c_1}^1, r_{c_1}^2)$ plays the role of $(r_{c_2}^1, r_{c_2}^2)$ in the proof above and we may continue to generate a sequence of couples, every second of which is unassigned in M . Again, the same proof above can be used to infer that if the number of couples is odd, then no stable matching can exist.

Conversely, we show that if the number of couples in I is even then I admits a stable matching. For ease of exposition we use the description of the instance I shown in Figure 9 for this part of the proof. For clarity, this instance is exactly

equivalent to the instance shown in Figure 8. Let M be the following matching in I where $h_{c_N}^{n_N+1} = h_{c_1}^0$ if $n_N > 0$ and $h_{c_N}^1 = h_{c_1}^0$ if $n_N = 0$:

$$\begin{aligned} M = & \{(r_{c_1}^1, h_{c_1}^0), (r_{c_1}^2, h_{c_1}^1)\} \\ & \cup \{(r_{c_k}^1, h_{c_{k-1}}^{n_{k-1}+1}), (r_{c_k}^2, h_{c_k}^1) : 2 \leq k \leq N, n_{k-1} > 0, k \bmod 2 \neq 0\} \\ & \cup \{(r_{c_k}^1, h_{c_{k-1}}^2), (r_{c_k}^2, h_{c_k}^1) : 2 \leq k \leq N, n_{k-1} = 0, k \bmod 2 \neq 0\} \\ & \cup \{(r_{s_k}^a, h_{c_k}^{a+1}) : 1 \leq k \leq N, 1 \leq a \leq n_k, n_k > 0\} \end{aligned}$$

Assume M is unstable. Then there must exist a blocking pair of M in I .

Clearly no single resident $r_{s_k}^a$ ($1 \leq k \leq N, 1 \leq a \leq n_k, n_k > 0$) can form part of a blocking pair for M in I as he is assigned in M to his first preference. Further, no couple $(r_{c_k}^1, r_{c_k}^2)$ ($2 \leq k \leq N, k \bmod 2 \neq 0$) can form part of a blocking pair for M in I as they are assigned to the hospital pair in first place on their joint preference list, $(h_{c_{k-1}}^{n_{k-1}+1}, h_{c_k}^1)$ if $n_k > 0$ or $(h_{c_{k-1}}^2, h_{c_k}^1)$ if $n_k = 0$.

Now, assume that $(r_{c_k}^1, r_{c_k}^2)$ ($2 \leq k \leq N, k \bmod 2 = 1$) blocks M . If $n_{k-1} > 0$ then $(r_{c_k}^1, r_{c_k}^2)$ blocks with $(h_{c_{k-1}}^{n_{k-1}+1}, h_{c_k}^1)$. However, $h_{c_k}^1$ is assigned in M to its first preference $r_{s_k}^1$ and so cannot form part of a blocking pair, a contradiction. If $n_k = 0$ then $(r_{c_k}^1, r_{c_k}^2)$ blocks M with $(h_{c_{k-1}}^2, h_{c_k}^1)$. However, $h_{c_k}^1$ is assigned in M to its first preference (either $r_{s_k}^1$ if $n_k > 0$, or $r_{c_{k+1}}^1$ if $n_k = 0$) and so cannot form part of a blocking pair, a contradiction. Since no other possible blocking pairs exist for M in I it must be the case that M is a stable matching in I and the result is proven. \square

Lemmas 12 and 13 lead to the following conclusion.

Theorem 14 $(2, 1, 2)$ -MIN BP HRC is solvable in polynomial time.

Proof. Let I be an instance of $(2, 1, 2)$ -HRC, and assume that M_0 is a matching in I in which all fixed assignments have been iteratively satisfied, and assume that the corresponding deletions have been made from the preference lists in I , yielding instance I' . Lemma 12 shows that I' is a union of sub-instances I_1, I_2, \dots, I_t , where each I_j is of the form shown in Figure 8 ($1 \leq j \leq t$).

For each j ($1 \leq j \leq t$), we show how to construct a matching M_j in sub-instance I_j such that $|bp(M_j)| \leq 1$. Let N be the number of couples in I_j . Suppose firstly that N is even. The proof of Lemma 13 shows how to construct a matching M_j that is stable in I_j .

Now suppose that N is odd. Then $N \geq 3$, since all fixed assignments in I have been iteratively satisfied. By Lemma 13, I_j does not admit a stable matching; we will construct a matching M_j in I such that $|bp(M_j)| = 1$. Let k ($1 \leq k \leq N$) be given.

Firstly assume that k is odd and $k \neq N$. Match $(r_{c_k}^1, r_{c_k}^2)$ to the hospital pair on their list. If $n_k > 0$, match $r_{s_k}^i$ to his first-choice hospital $h_{c_k}^{i+1}$ ($1 \leq i \leq n_k$). Now assume that k is even. Leave $(r_{c_k}^1, r_{c_k}^2)$ unassigned. If $n_k > 0$, match $r_{s_k}^i$ to his second-choice hospital $h_{c_k}^i$ ($1 \leq i \leq n_k$). Finally assume that $k = N$. If $n_N = 0$, leave couple $(r_{c_N}^1, r_{c_N}^2)$ unassigned. Otherwise match $(r_{c_N}^1, r_{c_N}^2)$ to the hospital pair on their list. Also for each i ($1 \leq i \leq n_N - 1$), match $r_{s_N}^i$ to his first-choice hospital $h_{c_N}^{i+1}$, and leave $r_{s_N}^{n_N}$ unassigned.

It is straightforward to verify that if $n_N > 0$ then $bp(M_j) = \{(r_{s_N}^{n_N}, h_{c_N}^{n_N})\}$ in I_j . Otherwise if $n_N = 0$ and $n_{N-1} = 0$, the only blocking pair of M_j in I_j involves the couple $(r_{c_{N-1}}^1, r_{c_{N-1}}^2)$ and the hospital pair $(h_{c_{N-2}}^1, h_{c_{N-1}}^1)$. Finally if $n_N = 0$ and $n_{N-1} > 0$, $bp(M_j) = \{(r_{s_{N-1}}^{n_{N-1}}, h_{c_{N-1}}^{n_{N-1}+1})\}$ in I_j .

Clearly $M = \cup_{j=0}^t M_j$ is then a most-stable matching in I . \square

D An Integer Programming formulation for MIN BP HRC

D.1 Introduction

The IP model for MIN BP HRC is based on modelling the various types of blocking pairs that might arise according to Definition 1, and allowing them to be counted by imposing a series of linear inequalities. The variables are defined for each resident, whether single or a member of a couple, and for each element on his preference list (with the possibility of being unassigned). A further consistency constraint ensures that each member of a couple obtains hospitals from the same pair in their list, if assigned. A suitable objective function then enables the number of blocking pairs to be minimised. Subject to this, we may also maximise the size of the constructed matching.

A crucial component of the IP model is a mapping between the joint preference list of a couple (r_i, r_j) and individual preference lists for r_i and r_j ; we call these individual lists the *projected preference lists* for r_i and r_j . Let I be an instance of HRC with residents $R = \{r_1, r_2, \dots, r_{n_1}\}$ and hospitals $H = \{h_1, h_2, \dots, h_{n_2}\}$. Without loss of generality, suppose residents $r_1, r_2 \dots r_{2c}$ are in couples. Again, without loss of generality, suppose that the couples are (r_{2i-1}, r_{2i}) ($1 \leq i \leq c$). Suppose that the joint preference list of a couple $\mathcal{C}_i = (r_{2i-1}, r_{2i})$ is:

$$\mathcal{C}_i : (h_{\alpha_1}, h_{\beta_1}), (h_{\alpha_2}, h_{\beta_2}) \dots (h_{\alpha_l}, h_{\beta_l})$$

From this list we create the following *projected preference list* for resident r_{2i-1} :

$$r_{2i-1} : h_{\alpha_1}, h_{\alpha_2} \dots h_{\alpha_l}$$

and the following projected preference list for resident r_{2i} :

$$r_{2i} : h_{\beta_1}, h_{\beta_2} \dots h_{\beta_l}$$

Let $l(\mathcal{C}_i)$ denote the length of the preference list of \mathcal{C}_i and let $l(r_{2i-1})$ and $l(r_{2i})$ denote the lengths of the projected preference lists of r_{2i-1} and r_{2i} respectively. Clearly we have that $l(r_{2i-1}) = l(r_{2i}) = l(\mathcal{C}_i)$. A given hospital h_j may appear more than once in the projected preference list of a resident belonging to a couple $\mathcal{C}_i = (r_{2i-1}, r_{2i})$.

The single residents are $r_{2c+1}, r_{2c+2} \dots r_{n_1}$, where each such resident r_i , has a preference list of length $l(r_i)$ consisting of individual hospitals $h_j \in H$. Each hospital $h_j \in H$ has a preference list of individual residents $r_i \in R$ of length

$l(h_j)$. Further, each hospital $h_j \in H$ has capacity $c_j \geq 1$, the maximum number of residents to which it may be assigned.

We describe the variables, constraints and objective function in the IP model J for the MIN BP HRC instance I in Sections D.2, D.3 and D.4 respectively. The text in bold before the definition of a constraint shows the blocking pair type from Definition 1 to which the constraint corresponds. Finally in Section D.5 we present a proof of correctness for the IP model for MIN BP HRC.

D.2 Variables in the IP model

In J , for each i ($1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_i)$), define a variable $x_{i,p}$ such that

$$x_{i,p} = \begin{cases} 1 & \text{if } r_i \text{ is assigned to his } p^{\text{th}} \text{ choice hospital} \\ 0 & \text{otherwise.} \end{cases}$$

For $p = l(r_i) + 1$ define a variable $x_{i,p}$ whose intuitive meaning is that resident r_i is unassigned. Thus we also have that

$$x_{i,l(r_i)+1} = \begin{cases} 1 & \text{if } r_i \text{ is unassigned} \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = \{x_{i,p} : 1 \leq i \leq n_1, 1 \leq p \leq l(r_i) + 1\}$. For ease of exposition we define some additional notation. For given i ($1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_i)$), let $\text{pref}(r_i, p)$ denote the hospital at position p of r_i 's preference list (this is the projected preference list of r_i if $1 \leq i \leq 2c$). For each i ($1 \leq i \leq c$) and p ($1 \leq p \leq l(r_{2i-1})$), let $\text{pref}((r_{2i-1}, r_{2i}), p)$ denote the hospital pair at position p on the joint preference list of (r_{2i-1}, r_{2i}) .

Also, for an acceptable resident-hospital pair (r_i, h_j) , let $\text{rank}(h_j, r_i) = q$ denote the rank that hospital h_j assigns resident r_i , where $1 \leq q \leq l(h_j)$. Further, for each j ($1 \leq j \leq n_2$) and q ($1 \leq q \leq l(h_j)$) let the set $R(h_j, q)$ contain the resident-position pairs (r_i, p) such that r_i is assigned a rank of q ($1 \leq q \leq l(h_j)$) by h_j and h_j is at position p ($1 \leq p \leq l(r_i)$) in r_i 's preference list (or r_i 's projected preference list if r_i belongs to a couple). Hence:

$$R(h_j, q) = \{(r_i, p) \in R \times \mathbb{Z} : \text{rank}(h_j, r_i) = q \wedge 1 \leq p \leq l(r_i) \wedge \text{pref}(r_i, p) = h_j\}.$$

Now, for all j ($1 \leq j \leq n_2$) and q ($1 \leq q \leq l(h_j)$), define a new variable $\alpha_{j,q} \in \{0, 1\}$. The intuitive meaning of a variable $\alpha_{j,q}$ is that if h_j is fully subscribed with assignees better than rank q then $\alpha_{j,q}$ may take the value 0 or 1. However, if h_j is not full with assignees better than rank q then $\alpha_{j,q} = 1$. Constraints (2) and 13 described in Section D.3 are applied to enforce this property.

Now, for all j ($1 \leq j \leq n_2$) and q ($1 \leq q \leq l(h_j)$), define a new variable $\beta_{j,q} \in \{0, 1\}$. The intuitive meaning of a variable $\beta_{j,q}$ is that if h_j has $c_j - 1$ or more assignees better than rank q then $\beta_{j,q}$ may take a value of 0 or 1. However, if h_j has fewer than $c_j - 1$ assignees better than rank q then $\beta_{j,q} = 1$. Constraints (3) and 14 described in Section D.3 are applied to enforce this property.

Finally, for all i ($1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_i)$), define a new variable $\theta_{i,p} \in \{0, 1\}$. The intuitive meaning of a variable $\theta_{i,p}$ is that $\theta_{i,p} = 1$ if resident r_i is involved in a blocking pair with the hospital at position p on his preference list, either as a single resident or as part of a couple, and $\theta_{i,p} = 0$ otherwise.

D.3 Constraints in the IP model

The following constraint simply ensures that each variable $x_{i,p}$ must be binary valued for all i ($1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_i) + 1$):

$$x_{i,p} \in \{0, 1\} \quad (1)$$

Similarly, the following constraint ensures that each variable $\alpha_{j,q}$ must be binary valued for all j ($1 \leq j \leq n_2$) and q ($1 \leq q \leq l(h_j)$):

$$\alpha_{j,q} \in \{0, 1\} \quad (2)$$

Also, the following constraint ensures that each variable $\beta_{j,q}$ must be binary valued for all j ($1 \leq j \leq n_2$) and q ($1 \leq q \leq l(h_j)$):

$$\beta_{j,q} \in \{0, 1\} \quad (3)$$

Similarly the following constraint ensures that each variable $\theta_{i,p}$ must be binary valued for all i ($1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_i) + 1$):

$$\theta_{i,p} \in \{0, 1\} \quad (4)$$

As each resident $r_i \in R$ is assigned to exactly one hospital or is unassigned (but not both), we introduce the following constraint for all i ($1 \leq i \leq n_1$):

$$\sum_{p=1}^{l(r_i)+1} x_{i,p} = 1 \quad (5)$$

Since a hospital h_j may be assigned at most c_j residents, $x_{i,p} = 1$ where $\text{pref}(r_i, p) = h_j$ for at most c_j residents. We thus obtain the following constraint for all j ($1 \leq j \leq n_2$):

$$\sum_{i=1}^{n_1} \sum_{p=1}^{l(r_i)} \{x_{i,p} \in X : \text{pref}(r_i, p) = h_j\} \leq c_j \quad (6)$$

For each couple (r_{2i-1}, r_{2i}) , r_{2i-1} is unassigned if and only if r_{2i} is unassigned, and r_{2i-1} is assigned to the hospital in position p in their individual list if and only if r_{2i} is assigned to the hospital in position p in their individual list. We thus obtain the following constraint for all i ($1 \leq i \leq c$) and p ($1 \leq p \leq l(r_{2i-1}) + 1$):

$$x_{2i-1,p} = x_{2i,p} \quad (7)$$

Type 1 blocking pairs. In matching M in I , if a single resident $r_i \in R$ is unassigned or has a worse partner than some hospital $h_j \in H$ where $\text{pref}(r_i, p) = h_j$ and $\text{rank}(h_j, r_i) = q$ then h_j must be fully subscribed with better partners than r_i , for otherwise (r_i, h_j) blocks M . Hence if r_i is unassigned or has worse partner than h_j , i.e., $\sum_{p'=p+1}^{l(r_i)+1} x_{i,p'} = 1$, and h_j is not fully subscribed with better partners than r_i , i.e., $\sum_{q'=1}^{q-1} \{x_{i',p''} \in X : (r_{i'}, p'') \in R(h_j, q')\} < c_j$, then we require $\theta_{i,p} = 1$ to count this blocking pair. Thus, for each i ($2c+1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_i)$) we obtain the following constraint where $\text{pref}(r_i, p) = h_j$ and $\text{rank}(h_j, r_i) = q$:

$$c_j \left(\left(\sum_{p'=p+1}^{l(r_i)+1} x_{i,p'} \right) - \theta_{i,p} \right) \leq \sum_{q'=1}^{q-1} \{x_{i',p''} \in X : (r_{i'}, p'') \in R(h_j, q')\} \quad (8)$$

In this way, we can count the number of blocking pairs using the $\theta_{i,p}$ values. A similar methodology is used in all replacement constraints for the remaining stability criteria that follow. Ultimately, the number of blocking pairs is the sum of the $\theta_{i,p}$ values, except that to avoid counting a blocking pair twice in the case of a couple, the model will assume that $\theta_{2i,p} = 0$ for all i ($1 \leq i \leq c$) and for all p ($1 \leq p \leq l(r_{2i})$).

Type 2a blocking pairs. In a matching M in I , if a couple $\mathcal{C}_i = (r_{2i-1}, r_{2i})$ jointly prefer hospital pair (h_{j_1}, h_{j_2}) , at position p_1 in \mathcal{C}_i 's joint preference list, to $(M(r_{2i-1}), M(r_{2i}))$, at position p_2 , and h_{j_1} is undersubscribed or prefers r_{2i-1} to one of its assignees in M , and $h_{j_2} = M(r_{2i})$, then (r_{2i-1}, r_{2i}) blocks M with (h_{j_1}, h_{j_2}) . In the special case where $\text{pref}(r_{2i-1}, p_1) = \text{pref}(r_{2i}, p_1) = h_{j_1}$, if $h_{j_1} = h_{j_2} = M(r_{2i})$, h_{j_1} is undersubscribed or prefers r_{2i-1} to one of its assignees in M other than r_{2i} , then again (r_{2i-1}, r_{2i}) blocks M with (h_{j_1}, h_{j_2}) .

Thus, for the general case, we obtain the following constraint for all i ($1 \leq i \leq c$) and p_1, p_2 ($1 \leq p_1 < p_2 \leq l(r_{2i-1})$) such that $\text{pref}(r_{2i}, p_1) = \text{pref}(r_{2i}, p_2)$ and $\text{rank}(h_{j_1}, r_{2i-1}) = q$:

$$c_{j_1} (x_{2i-1,p_2} - \theta_{2i-1,p_1}) \leq \sum_{q'=1}^{q-1} \{x_{i',p''} \in X : (r_{i'}, p'') \in R(h_{j_1}, q')\} \quad (9)$$

For the special case in which $\text{pref}(r_{2i-1}, p_1) = \text{pref}(r_{2i}, p_1) = h_{j_1}$ we obtain the following constraint for all i ($1 \leq i \leq c$) and p_1, p_2 where ($1 \leq p_1 < p_2 \leq l(r_{2i-1})$) such that $\text{pref}(r_{2i}, p_1) = \text{pref}(r_{2i}, p_2)$ and $\text{rank}(h_{j_1}, r_{2i-1}) = q$:

$$(c_{j_1} - 1)(x_{2i-1,p_2} - \theta_{2i-1,p_1}) \leq \sum_{q'=1}^{q-1} \{x_{i',p''} \in X : q' \neq \text{rank}(h_{j_1}, r_{2i}) \wedge (r_{i'}, p'') \in R(h_{j_1}, q')\} \quad (10)$$

Type 2b blocking pairs. A similar constraint is required for the case that the odd-subscript member of a given couple stays assigned to the same hospital. Thus, for the general case, we obtain the following constraint for all i ($1 \leq i \leq c$) and p_1, p_2 where ($1 \leq p_1 < p_2 \leq l(r_{2i})$) such that $\text{pref}(r_{2i-1}, p_1) = \text{pref}(r_{2i-1}, p_2)$ and $\text{rank}(h_{j_2}, r_{2i}) = q$:

$$c_{j_2}(x_{2i-1, p_2} - \theta_{2i-1, p_1}) \leq \sum_{q'=1}^{q-1} \{x_{i', p''} \in X : (r_{i'}, p'') \in R(h_{j_2}, q')\} \quad (11)$$

Again, for the special case in which $\text{pref}(r_{2i-1}, p_1) = \text{pref}(r_{2i}, p_1) = h_{j_2}$ we obtain the following constraint for all i ($1 \leq i \leq c$) and p_1, p_2 where ($1 \leq p_1 < p_2 \leq l(r_{2i})$) such that $\text{pref}(r_{2i-1}, p_1) = \text{pref}(r_{2i-1}, p_2)$ and $\text{rank}(h_{j_2}, r_{2i}) = q$:

$$(c_{j_1} - 1)(x_{2i-1, p_2} - \theta_{2i-1, p_1}) \leq \sum_{q'=1}^{q-1} \{x_{i', p''} \in X : q' \neq \text{rank}(h_{j_2}, r_{2i-1}) \wedge (r_{i'}, p'') \in R(h_{j_2}, q')\} \quad (12)$$

Now, we define a variable $\alpha_{j,q}$ such that if h_j is full with assignees better than rank q then $\alpha_{j,q}$ may take the value of zero or one. Otherwise, h_j is not full with assignees better than rank q and $\alpha_{j,q} = 1$. Hence, we obtain the following constraint for all j ($1 \leq j \leq n_2$) and q ($1 \leq q \leq l(h_j)$):

$$\alpha_{j,q} \geq 1 - \frac{\sum_{q'=1}^{q-1} \{x_{i,p} \in X : (r_i, p) \in R(h_j, q')\}}{c_j} \quad (13)$$

Next we define a variable $\beta_{j,q}$ such that if h_j has $c_j - 1$ or more assignees better than rank q then $\beta_{j,q}$ may take a value of zero or one. Otherwise, h_j has fewer than $c_j - 1$ assignees better than rank q and $\beta_{j,q} = 1$. Hence, we obtain the following constraint all j ($1 \leq j \leq n_2$) and q ($1 \leq q \leq l(h_j)$):

$$\beta_{j,q} \geq 1 - \frac{\sum_{q'=1}^{q-1} \{x_{i,p} \in X : (r_i, p) \in R(h_j, q')\}}{(c_j - 1)} \quad (14)$$

Type 3a blocking pairs. In a matching M in I , if a couple $\mathcal{C}_i = (r_{2i-1}, r_{2i})$ is unassigned or assigned to a worse hospital pair than (h_{j_1}, h_{j_2}) (where $h_{j_1} \neq h_{j_2}$), and for each $t \in \{1, 2\}$, h_{j_t} is undersubscribed and finds r_{2i-2+t} acceptable, or prefers r_{2i-2+t} to its worst assignee, then (r_{2i-1}, r_{2i}) blocks M with (h_{j_1}, h_{j_2}) . Thus we obtain the following constraint for all i ($1 \leq i \leq c$) and p ($1 \leq p \leq l(r_{2i-1})$) where $h_{j_1} = \text{pref}(r_{2i-1}, p)$, $h_{j_2} = \text{pref}(r_{2i}, p)$, $h_{j_1} \neq h_{j_2}$, $\text{rank}(h_{j_1}, r_{2i-1}) = q_1$ and $\text{rank}(h_{j_2}, r_{2i}) = q_2$:

$$\sum_{p'=p+1}^{l(r_{2i-1})+1} x_{2i-1, p'} + \alpha_{j_1, q_1} + \alpha_{j_2, q_2} - \theta_{2i-1, p} \leq 2 \quad (15)$$

Type 3b/c blocking pairs. In a matching M in I , if a couple $\mathcal{C}_i = (r_{2i-1}, r_{2i})$ is unassigned or assigned to a worse pair than (h_j, h_j) where $M(r_{2i-1}) \neq h_j$ and $M(r_{2i}) \neq h_j$, (r_{2i-1}, r_{2i}) finds (h_j, h_j) acceptable, and h_j has two or more free posts available, then (r_{2i-1}, r_{2i}) blocks M with (h_j, h_j) – this is a Type 3b blocking pair. In a matching M in I , if a couple $\mathcal{C}_i = (r_{2i-1}, r_{2i})$ is unassigned or assigned to a worse pair than (h_j, h_j) where $M(r_{2i-1}) \neq h_j$ and $M(r_{2i}) \neq h_j$, (r_{2i-1}, r_{2i}) finds (h_j, h_j) acceptable, and h_j prefers at least one of r_{2i-1} or r_{2i} to some assignee of h_j in M while simultaneously having a single free post, then (r_{2i-1}, r_{2i}) blocks M with (h_j, h_j) – this is a Type 3c blocking pair.

These two blocking pair types may be modelled by a single constraint. For each i ($1 \leq i \leq c$) and p ($1 \leq p \leq l(r_{2i-1})$) such that $\text{pref}(r_{2i-1}, p) = \text{pref}(r_{2i}, p)$ and $h_j = \text{pref}(r_{2i-1}, p)$, where $q = \min\{\text{rank}(h_j, r_{2i}), \text{rank}(h_j, r_{2i-1})\}$, we enforce the following:

$$c_j \left(\left(\sum_{p'=p+1}^{l(r_{2i-1})+1} x_{2i-1,p'} \right) - \theta_{2i-1,p} \right) - \frac{\sum_{q'=1}^{q-1} \{x_{i',p''} \in X : (r_{i'}, p'') \in R(h_j, q')\}}{(c_j - 1)} \leq \sum_{q'=1}^{l(h_j)} \{x_{i',p''} \in X : (r_{i'}, p'') \in R(h_j, q')\} \quad (16)$$

Type 3d blocking pairs. In a matching M in I , if a couple $\mathcal{C}_i = (r_{2i-1}, r_{2i})$ is unassigned or jointly assigned to a worse pair than (h_j, h_j) where $M(r_{2i-1}) \neq h_j$ and $M(r_{2i}) \neq h_j$, and h_j is full and also has two assignees r_s and r_t (where $s \neq t$) such that h_j prefers r_{2i-1} to r_s and h_j prefers r_{2i} to r_t , then (r_{2i-1}, r_{2i}) blocks M with (h_j, h_j) .

For each (h_j, h_j) acceptable to (r_{2i-1}, r_{2i}) , let r_{\min} be the better of r_{2i-1} and r_{2i} according to hospital h_j with $\text{rank}(h_j, r_{\min}) = q_{\min}$. Analogously, let r_{\max} be the worse of r_{2i} and r_{2i-1} according to hospital h_j with $\text{rank}(h_j, r_{\max}) = q_{\max}$. Then we obtain the following constraint for i ($1 \leq i \leq c$) and p ($1 \leq p \leq l(r_{2i-1})$) such that $\text{pref}(r_{2i-1}, p) = \text{pref}(r_{2i}, p) = h_j$:

$$\sum_{p'=p+1}^{l(r_{2i-1})+1} x_{2i-1,p'} + \alpha_{j,q_{\max}} + \beta_{j,q_{\min}} - \theta_{2i-1,p} \leq 2 \quad (17)$$

D.4 Objective function in the IP model

A maximum cardinality most-stable matching M is a matching of maximum cardinality, taken over all most-stable matchings in I . To compute such a matching in J , we apply two objective functions in sequence.

First we find an optimal solution in J that minimises the number of blocking pairs. To this end, we apply the following objective function:

$$\min \sum_{i=1}^{n_1} \sum_{p=1}^{l(r_i)} \theta_{i,p} \quad (18)$$

The matching M corresponding to an optimal solution in J will be a most-stable matching in I . Let $k = |bp(M)|$. Now we seek a maximum cardinality matching in I with at most k blocking pairs. Thus we add the following constraint to J , which ensures that, when maximising on cardinality, any solution also has at most k blocking pairs:

$$\sum_{i=1}^{n_1} \sum_{p=1}^{l(r_i)} \theta_{i,p} \leq k \quad (19)$$

The final step is to maximise the size of the matching, subject to the matching being most-stable. This involves optimising for a second time, this time using the following objective function:

$$\max \sum_{i=1}^{n_1} \sum_{p=1}^{l(r_i)} x_{i,p}. \quad (20)$$

D.5 Proof of correctness for the IP model

We now establish the correctness of the IP model for MIN BP HRC presented in Sections D.2, D.3 and D.4.

Theorem 15 *Given an instance I of MIN BP HRC, let J be the corresponding IP model as defined in Sections D.2, D.3 and D.4 (omitting Constraint (19) and objective function (20)). A most-stable matching in I is exactly equivalent to an optimal solution to J with respect to objective function (18).*

Proof. Let M be a most-stable matching in I . Let $\langle \mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$ be the corresponding assignment of boolean values to the variables in J as constructed in the proof of Theorem 12 in [7]. Initially let $\theta_{i,p} = 0$ for all i ($1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_i)$).

Assume that (r_i, h_j) blocks M where r_i is a single resident and $\text{pref}(r_i, p) = h_j$. Then Constraint (8) will be violated if $\theta_{i,p} = 0$. Set $\theta_{i,p} = 1$. Then the LHS of Constraint (8) becomes 0 and the constraint is satisfied.

Now, assume that (r_i, r_j) blocks M with (h_k, h_l) for some couple (r_i, r_j) , where $\text{pref}((r_i, r_j), p) = (h_k, h_l)$. Then depending on which part of Definition 1 is violated, one of the constraints in the range (9)-(12) and (15)-(17) will be violated if $\theta_{i,p} = 0$. By setting $\theta_{i,p} = 1$, the constraint concerned will be satisfied.

It follows that $\langle \mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta} \rangle$ is a feasible solution to J . Moreover the objective value of this solution is equal to $|bp(M)|$.

Conversely, let $\langle \mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta} \rangle$ be an optimal solution to J and let M be the corresponding matching in I as constructed in the proof of Theorem 12 in [7].

Now assume that $\theta_{i,p} = 1$ for some i ($1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_i)$). If r_i is not involved in a blocking pair with h_j where $\text{pref}(r_i, p) = h_j$ (either as a single resident or part of a couple), then by Theorem 12 in [7], Constraints (8) to (12) and (15)-(17) are satisfied with $\theta_{i,p} = 0$, in contradiction to the fact that $\langle \mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta} \rangle$ is optimal according to objective function (18). Thus if $\theta_{i,p} = 1$

for some i ($1 \leq i \leq n_1$) and p ($1 \leq p \leq l(r_{n_1})$) then r_i must be involved in a blocking pair with the hospital in position p on his preference list.

On the other hand, by the first direction, if there is a blocking pair of M , there must be a unique corresponding $\theta_{i,p}$ that has value 1. It follows that $|bp(M)|$ is equal to the objective value of $\langle \mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta} \rangle$ in J . \square

By enforcing Constraint (19) and imposing objective function (20), we obtain the following corollary.

Corollary 16 *Given an instance I of MIN BP HRC let J be the corresponding IP model as defined in Sections D.2, D.3 and D.4 (omitting objective function (18)). A maximum cardinality most-stable matching in I is exactly equivalent to an optimal solution to J with respect to objective function (20).*